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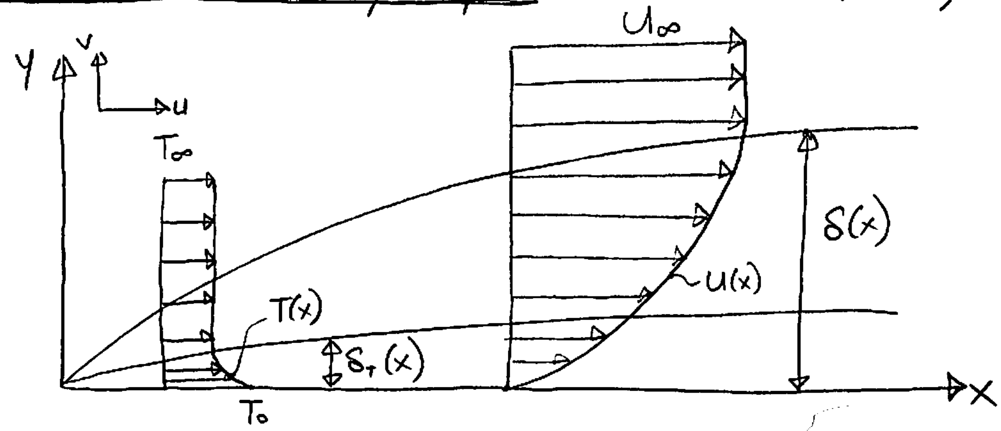
Thin Thermal Boundary Layer ($\delta \approx \delta_T$ or $\delta_T << \delta$)



In this limit, the thermal boundary layer is much thinner than the hydrodynamic boundary layer.

Here it gets a bit tricky. When scaling u in the energy equation, we must use the u in the thermal boundary layer, which in this case is not U_∞!

$$u \not\sim U_\infty$$

$$u \sim U_\infty \left(\frac{\delta_T}{\delta}\right) \quad \text{(Linear approximation} \Rightarrow \text{ok for scaling)}$$

Now we can deal with v:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{u}{L} + \frac{v}{\delta_T} = 0$$

$\Rightarrow$ Note here I used δ_T instead of δ. Again, this is due to δ_T being well within δ and the length scale being pertinent to heat trans.

$$v \sim u\left(\frac{\delta_T}{L}\right) \sim U_\infty \frac{\delta_T^2}{\delta L}$$

Now we can check our convective terms again to see the dominant one

$$u \frac{\Delta T}{L} \sim U_\infty \frac{\delta_T}{\delta} \cdot \frac{\Delta T}{L}$$

$$v \frac{\Delta T}{\delta_T} \sim U_\infty \frac{\delta_T^{\cancel{2}}}{\delta L} \cdot \frac{\Delta T}{\cancel{\delta_T}} \sim U_\infty \frac{\delta_T}{\delta} \frac{\Delta T}{L}$$

Both are important & must be kept

Nenad Miljkovic

Back into our energy equation:
convection ~ conduction

$$U_\infty \frac{\delta_T}{\delta} \frac{\cancel{\Delta T}}{L} \sim \alpha \frac{\cancel{\Delta T}}{\delta_T^2}$$

We know from our hydrodynamic solution that: $\frac{\delta}{L} \sim Re_L^{-1/2}$
Back substituting: (inertia ~ friction)

$$U_\infty \frac{\delta_T}{L\,Re_L^{-1/2}} \cdot \frac{1}{L} \sim \alpha \frac{1}{\delta_T^2} \quad ; \quad Pr = \frac{\nu}{\alpha} \Rightarrow \alpha = \frac{\nu}{Pr}$$

$$U_\infty \frac{\delta_T^3}{L^2 Re_L^{-1/2}} \sim \frac{\nu}{Pr}$$

$$\frac{\delta_T^3}{L^3} \sim \frac{\nu\, Re_L^{-1/2}}{Pr \cdot L\, U_\infty} \sim \underbrace{\left(\frac{\nu}{U_\infty L}\right)}_{Re_L^{-\frac{2}{2}}} Pr^{-1} Re_L^{-1/2}$$

$$\frac{\delta_T}{L} \sim \left(Pr^{-1} Re_L^{-3/2}\right)^{1/3}$$

$$\boxed{\frac{\delta_T}{L} \sim Pr^{-1/3} Re_L^{-1/2}} \quad \text{for } \frac{\delta_T}{\delta} \ll 1.$$

This means that:

$$\frac{\delta_T}{\delta} = \left(\frac{\delta_T}{L}\right) \cdot \left(\frac{L}{\delta}\right) \sim Pr^{-1/3} \cancel{Re_L^{-1/2}} \cdot \cancel{Re_L^{+1/2}}$$

$$\boxed{\frac{\delta_T}{\delta} \sim Pr^{-1/3} \ll 1} \Rightarrow \text{This is valid for fluids with } Pr^{1/3} \gg 1.$$

These include highly viscous fluids such as oils, honey, etc...
Solving for h & Nu:

$$\boxed{h \sim \frac{k}{\delta_T} \sim \frac{k}{L} Pr^{1/3} Re_L^{1/2}} \quad (Pr \gg 1)$$

$$\boxed{Nu = \frac{hL}{k} \sim Pr^{1/3} Re_L^{1/2}} \quad (Pr \gg 1)$$

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Note we could have done the scaling in terms of a timescale analysis:

Conduction $\Rightarrow$ $\tau_{\delta_T} \sim \dfrac{\delta_T^2}{\alpha}$ (from $D \sim \sqrt{\alpha t}$)

Convection $\Rightarrow$ $\tau_L \sim \dfrac{L}{U_\infty}$ (for $Pr < 1$) Longitudinal speed $\sim U_\infty$

$\tau_L \sim \dfrac{L}{\frac{\delta_T}{\delta} U_\infty}$ (for $Pr > 1$) Longitudinal speed $\sim \frac{\delta_T}{\delta} U_\infty$

Scaling our timescales with eachother:

<u>$Pr < 1$:</u>

$$\tau_{\delta_T} \sim \tau_L \Rightarrow \frac{\delta_T^2}{\alpha} \sim \frac{L}{U_\infty} \qquad \Rightarrow Pe = \frac{U_\infty L}{\alpha}$$

$$\boxed{\delta_T \sim \sqrt{\frac{\alpha L}{U_\infty}} \sim L \cdot Pe^{-1/2}} \Rightarrow \text{Same as before.}$$

<u>$Pr > 1$:</u>

$$\tau_{\delta_T} \sim \tau_L \Rightarrow \frac{\delta_T^2}{\alpha} \sim \frac{L}{\frac{\delta_T}{\delta} U_\infty}$$

$$\delta_T^3 \sim \frac{\delta L \alpha}{U_\infty} \Rightarrow \delta \sim L \cdot Re_L^{-1/2}$$

$$\delta_T^3 \sim \frac{L^2 \alpha Re_L^{-1/2}}{U_\infty} \sim \frac{L^2 \nu}{U_\infty Pr} \cdot Re_L^{-1/2}$$

$$\boxed{\delta_T \sim L \cdot Pr^{-1/3} Re_L^{-1/2}} \Rightarrow \text{Same as before.}$$

<u>Meaning of Reynolds Number</u>

For external flows, $Re_L = U_\infty L/\nu$. For all of your lives you've been told that Re_L = Inertia/Friction. But we just said that for laminar b.l. Inertia $\sim$ Friction, up to $10^5 = Re_L$. The way to interpret Reynolds number in b.l. flow is geometric:

$$\boxed{Re_L^{1/2} = \frac{L}{\delta} = \frac{\text{wall length}}{\text{b.l. thickness}}} \Rightarrow \text{sometimes called "slenderness" ratio}$$

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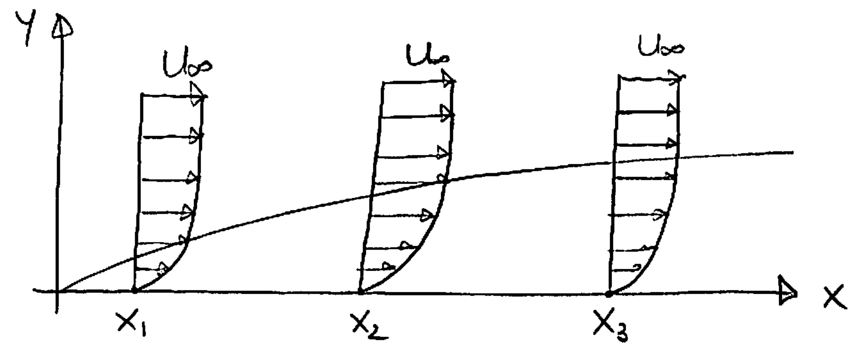
Similarity Solutions

The basic idea to these solutions is the observation that from one location x to another, the u & T profiles look similar. For example:



For all three locations, we know that $u(x=0)=0$
$u(x\to\infty)=U_\infty$

From this, we can argue that:

$$\frac{u}{U_\infty} = f'(\eta) \Rightarrow \eta \equiv \text{similarity variable.}$$

We can intuitively say that $\eta \sim y$, and $\eta = f(x)$.
From our previous scaling analysis, we can deduce:

$$\eta \sim \frac{y}{\delta(x)} \sim \frac{y}{x\,Re_x^{-1/2}}$$

So we can assume: $$\boxed{\eta = \frac{y}{x} Re_x^{1/2} = y\sqrt{\frac{U_\infty}{\nu x}}}$$ $\Rightarrow$ Similarity Variable.

We also assume that:

$$\boxed{\frac{u}{U_\infty} = f'(\eta)}$$

Our equations to work with are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Back substituting and solving for idividual terms:

$$\frac{\partial u}{\partial x} = U_\infty \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = U_\infty f'' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu}} \cdot x^{-1/2}\right) = y\sqrt{\frac{U_\infty}{\nu}} \cdot -\frac{1}{2}x^{-3/2} = -\frac{1}{2x}\eta$$

$$\frac{\partial u}{\partial x} = -U_\infty f'' \frac{1}{2x}\eta$$

Now we can use continuity to solve for v.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = U_\infty f'' \frac{1}{2x}\eta$$

$$\frac{v}{U_\infty} = \frac{1}{2x}\int_0^y f''\eta \, dy = \underbrace{\frac{1}{2x}\sqrt{\frac{x\nu}{U_\infty}}}_{\frac{1}{2} \cdot Re_x^{-1/2}} \int_0^\eta f''\eta \, d\eta \quad \left(\text{since } dy = \sqrt{\frac{x\nu}{U_\infty}}\, d\eta\right)$$

Now we can solve the above integral

$$\int_0^\eta \eta f'' d\eta = uv - \int v\,du \quad \left(IBP: \int u\,dv = uv - \int v\,du\right)$$

here $u = \eta$, $v = f'$, $dv = f'' d\eta$

$$\int_0^\eta \eta f'' d\eta = \eta f' - \int_0^\eta f' d\eta$$

$$= \eta f' - f$$

So: $$\frac{v}{U_\infty} = \frac{1}{2} Re_x^{-1/2}(\eta f' - f)$$

Now we can solve for our last term:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(U_\infty \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial y}\left(U_\infty f'' \frac{Re_x^{1/2}}{x}\right)$$

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$$= U_\infty \frac{Re_x^{1/2}}{x} \frac{\partial}{\partial y}(f'') = \frac{U_\infty Re_x^{1/2}}{x} \frac{\partial f''}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty Re_x}{x^2} f'''$$

Back substituting all of our terms into our momentum eqn.

$$\cancel{U_\infty} f'\left(-U_\infty f'' \frac{1}{2\cancel{x}}\eta\right) + \cancel{U_\infty}\frac{1}{2} Re_x^{-1/2}(\eta f' - f)\left(U_\infty f'' \frac{Re_x^{1/2}}{\cancel{x}}\right)$$

$$= \nu \frac{\cancel{U_\infty} Re_x}{x^{\cancel{2}}} f'''$$

$$\cancel{-f'f'' U_\infty \frac{\eta}{2}} + \cancel{\frac{1}{2} Re_x^{-1/2} \eta f' U_\infty f'' Re_x^{1/2}} - \frac{1}{2} Re_x^{-1/2} f U_\infty f'' Re_x^{1/2}$$

$$- \nu \frac{Re_x}{x} f''' = 0$$

$$-\frac{1}{2} f U_\infty f'' - \nu \frac{Re_x}{x} f''' = 0$$

We know $Re_x = \frac{U_\infty x}{\nu}$, back substituting

$$\frac{1}{2} f \cancel{U_\infty} f'' + \cancel{\nu} \frac{\cancel{U_\infty} \cancel{x}}{\cancel{x}\cancel{\nu}} f''' = 0$$

$$\boxed{\frac{1}{2} f f'' + f''' = 0} \quad , \quad \boxed{f'(\eta) = \frac{u}{U_\infty}} \quad \boxed{\eta = y\sqrt{\frac{U_\infty}{\nu x}}}$$

Now we've converted our PDE into an ODE
Converting our boundary conditions:

$$u(y=0) = 0 \Rightarrow \eta(y=0) = 0 \ ; \ f'(0) = 0$$

$$u(y\to\infty) = U_\infty \Rightarrow \eta(y\to\infty) = \infty \ ; \ f'(\eta\to\infty) = 1$$

$$v(y=0) = 0 \Rightarrow \eta(y=0) = 0 \ ; \quad v = U_\infty \frac{1}{2} Re_x^{-1/2}(\eta(0)\cdot f'(0) - f(0))$$

$$f(0) = 0$$

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To solve, we can assume an infinite series solution:

$$
\begin{aligned}
f &= a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots + \\
f' &= a_1 + 2a_2\eta + 3a_3\eta^2 + \cdots + \\
f'' &= 2a_2 + 6a_3\eta + \cdots + \\
f''' &= 6a_3 + \cdots +
\end{aligned}
$$

Back substituting and solving for our coefficients (matrix form)

$$\underbrace{(\cdots)}_{0}\eta^0 + \underbrace{(\cdots)}_{0}\eta^1 + \underbrace{(\cdots)}_{0}\eta^2 + \underbrace{(\cdots)}_{0}\eta^3 + \cdots + \underbrace{(\cdots)}_{0}\eta^n = 0$$

We can obtain a a recursion formula relating our constants

$$\boxed{f = \frac{a_2\eta^2}{2!} - \frac{a_2\eta^5}{2\cdot 5!} + \frac{11}{4}\frac{a_3^{\,3}\eta^8}{8!} + \cdots}$$ ⇒ Blasius Solution (1911) Prandtl's PhD Student

$\boxed{a_2 = 0.332}$ ⇒ since $f(0) = f'(0) = 0$ ⇒ $\boxed{a_1 = a_0 = 0}$

We know that $f'(\eta) = \frac{u}{U_\infty}$

$$f'(\eta) = \frac{\cancel{2}a_2\eta}{\cancel{2}} + \frac{5\cdot a_2\eta^4}{2\cdot 120} + \left(\frac{11}{4}\right)\frac{8a_3^{\,3}\eta^7}{40320} + \cdots$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

Looking at only our first term: $\eta \to 0$ (higher order terms drop)

$$f'(\eta) = \frac{u}{U_\infty} = 0.332\, y\sqrt{\frac{U_\infty}{\nu x}} \quad \text{or} \quad \boxed{u(x,y) = 0.332\, y\sqrt{\frac{U_\infty^3}{\nu x}}}$$

Note we can solve numerically and plot our non-dimensional result. Also, the above $u(x,y)$ result is only valid near the wall where $\eta \ll 1$, so h.o.T drop out.

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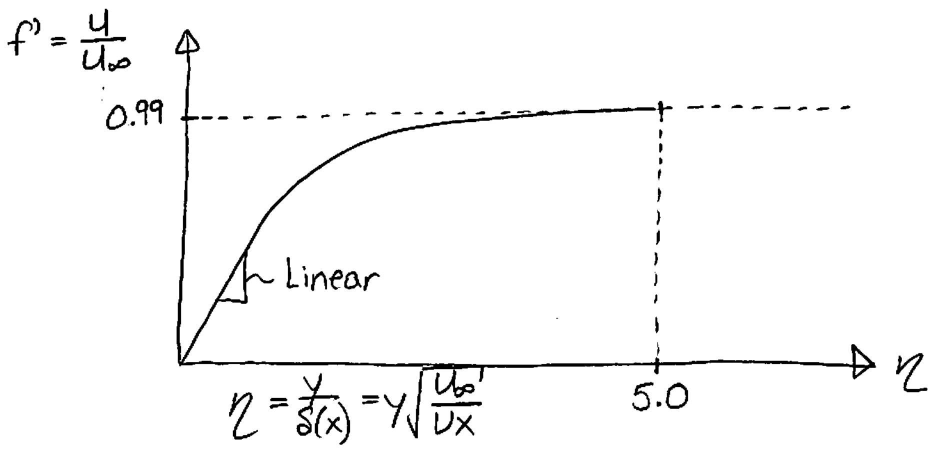

Solving for when $f' = 0.99$ (when $u = 0.99U_\infty \equiv$ b.l. thickness)

$$5.0 = \delta\sqrt{\frac{U_\infty}{\nu x}} = \frac{\delta}{x}Re_x^{1/2}$$

$$\boxed{\delta = \frac{5x}{\sqrt{Re_x}}}$$ $\Rightarrow$ Hydrodynamic boundary layer thickness on a flat plate in laminar flow.

Now we can use this super usefull information to do some calculations:

$$\tau(x) = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} \Rightarrow \text{we know } f' = \overbrace{0.332}^{a_2}\,y\sqrt{\frac{U_\infty}{\nu x}} = \frac{u}{U_\infty} \text{ at } \eta \to 0$$

$$u = 0.332\,U_\infty\,y\sqrt{\frac{U_\infty}{\nu x}}$$ $\Rightarrow$ only the first term from our solution since $\eta \ll 1$, and higher order terms drop out.

$$\frac{\partial u}{\partial y}\bigg|_{y=0} = 0.332\,U_\infty\sqrt{\frac{U_\infty}{\nu x}}$$

$$\tau(x) = 0.332\,U_\infty\,\mu\sqrt{\frac{U_\infty\rho}{\mu x}} = 0.332\,U_\infty\sqrt{\frac{\mu U_\infty\rho}{x}}\cdot\left(\frac{\rho U_\infty}{\rho U_\infty}\right)\left(\frac{2}{2}\right)$$

$$= 0.664\cdot\frac{1}{2}\rho U_\infty^2\sqrt{\frac{\mu \cancel{U_\infty}\cancel{\rho}}{\rho^{\cancel{2}}U_\infty^{\cancel{2}}x}} = 0.664\cdot\frac{1}{2}\cdot\rho U_\infty^2\cdot Re_x^{-1/2}$$

$$\boxed{\frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = 0.664\cdot Re_x^{-1/2} = C_{f,x}}$$ $\Rightarrow$ Skin friction coefficient

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Typically, we want the averaged friction coefficient:

$$\overline{\tau} = \frac{1}{L}\int_0^L \tau(x)\,dx$$

Noting that $\tau(x) = C x^{-1/2}$, $C = \frac{1}{2}\rho U_\infty^2 \cdot (0.664)\, U_\infty^{1/2} \nu^{-1/2}$

$$\overline{\tau} = C \cdot \frac{1}{L}\int_0^L \frac{dx}{x^{1/2}} = \frac{2C}{L^{1/2}}$$

$\boxed{\overline{\tau} = 0.664\,\rho U_\infty^2 \cdot Re_L^{-1/2}}$ ⇒ Wall averaged shear

$\boxed{\overline{C_f} = \frac{\overline{\tau}}{\frac{1}{2}\rho U_\infty^2} = \frac{1.328}{Re_L^{1/2}}}$ ⇒ Wall averaged skin friction coefficient.

Note on page 41 of the notes, we solved from similarity:

$$\left.\begin{aligned} \overline{\tau} &\sim \rho U_\infty^2 Re_L^{-1/2} \\ \overline{C_f} &\sim Re_L^{-1/2} \end{aligned}\right\}$$ Pretty good when considering the ease in which we got these.

Example | Calculate the boundary layer thickness at the windshield of a car moving at 70 mph.

U_∞ = 70 mph (31.11 m/s)
$\nu_{air} = 1.5 \times 10^{-5}\ m^2/s$
L = 1 m (length from the front bumper to the windshield)

$$Re_L = \frac{V_\infty L}{\nu} = \frac{(31.11 m/s)(1m)}{(1.5\times10^{-5} m/s)} \cong 2.07\times10^6$$

$$\delta \cong \frac{5L}{\sqrt{Re_L}} = \frac{5m}{\sqrt{2.07\times10^6}} = 0.0035m = 3.5mm$$

$\boxed{\delta = 3.5mm}$ ⇒ Very thin!

No wonder Prandtl had a hard time seeing it!