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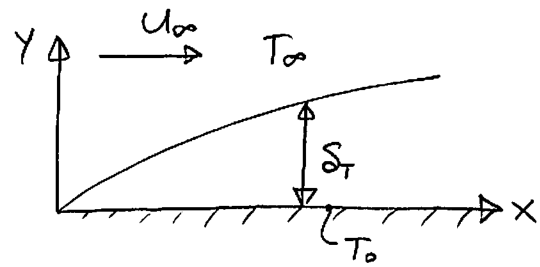
Heat Transfer (Similarity Solution)

$$h = \frac{q''|_{y=0}}{\Delta T} = \frac{q''|_{y=0}}{T_0 - T_\infty} = ?$$



Looking at our energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$T(y=0) = T_0$$
$$T(y\to\infty) = T_\infty$$

Non dimensionalizing: $\theta = \dfrac{T - T_0}{T_\infty - T_0}$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \iff u\frac{\partial \bar{u}}{\partial x} + v\frac{\partial \bar{u}}{\partial y} = \nu \frac{\partial^2 \bar{u}}{\partial y^2}$$

B.L. momentum equation

$$\theta(y=0) = 0$$
$$\theta(y\to\infty) = 1$$
$$\left.\frac{\partial \theta}{\partial y}\right|_{y\to\infty} = 0$$

Noting that a usefull connection between the energy & hydrodynam equations is $Pr = \nu/\alpha$

If $Pr = 1$, $\alpha = \nu$ and $\theta = \bar{u}$ and $\delta = \delta_T$

Usually however, $Pr \neq 1$, so we have to solve for θ

Assuming an identical similarity variable as before

$$\eta = y\sqrt{\frac{U_\infty}{x\nu}}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial \theta}{\partial \eta}\cdot\left(-\frac{y}{2}\sqrt{\frac{U_\infty}{\nu x^3}}\right)$$

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$$\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial \eta}\right) \cdot \left(\frac{\partial \eta}{\partial y}\right) + \cancel{\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y}\right)}\left(\frac{\partial \theta}{\partial y}\right)^{0}$$

$$= \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\left(\frac{\partial \theta}{\partial \eta}\right) \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 \theta}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2$$

Back substituting into our energy equation:

$$u \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + v \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y}\right)^2 \quad (1)$$

To solve, we need to use our hydrodynamic b.l. definitions:
On page 48 of notes, we already solved that:

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2x} \cdot \eta$$

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{x\nu}}$$

$$u = U_\infty f' \quad \left(\text{Definition, we defined } f' = \frac{u}{U_\infty}\right)$$

$$v = U_\infty \cdot \frac{1}{2x}\sqrt{\frac{x\nu}{U_\infty}}\left(\eta f' - f\right) \quad (\text{From continuity})$$

Back substituting into equation (1)

$$-\cancel{U_\infty} f' \frac{\partial \theta}{\partial \eta} \cdot \underbrace{\frac{y}{\cancel{2x}}\sqrt{\frac{U_\infty}{x\nu}}}_{\frac{\eta}{2}} + \frac{\cancel{U_\infty}}{\cancel{2x}}\sqrt{\frac{x\nu}{U_\infty}}\left(\eta f' - f\right)\frac{\partial \theta}{\partial \eta}\sqrt{\frac{U_\infty}{x\nu}} = \alpha \frac{\partial^2 \theta}{\partial \eta^2} \cdot \frac{\cancel{U_\infty}}{\cancel{x}\nu}$$

$$-f' \frac{\partial \theta}{\partial \eta} \cdot \frac{\eta}{2} + f' \frac{\partial \theta}{\partial \eta} \cdot \frac{\eta}{2} \cdot \cancel{\sqrt{\frac{x\nu}{U_\infty}}} \cdot \cancel{\sqrt{\frac{U_\infty}{x\nu}}} - \frac{1}{2} f \frac{\partial \theta}{\partial \eta} \cancel{\sqrt{\frac{x\nu}{U_\infty}}} \cdot \cancel{\sqrt{\frac{U_\infty}{x\nu}}}$$

$$= \alpha \frac{\partial^2 \theta}{\partial \eta^2} \frac{1}{\nu}$$

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The first two terms cancel and we are left with:

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2}\underbrace{\frac{\nu}{\alpha}}_{Pr} f \frac{\partial \theta}{\partial \eta} = 0$$

$$\boxed{\theta'' + \frac{1}{2} Pr \cdot f \cdot \theta' = 0} \quad (2)$$

We've turned our energy PDE into an ODE
To solve we can integrate but here we can use a trick:
Usually most books do the following:

$$\frac{d\theta'}{d\eta} + \frac{1}{2} Pr \cdot f \cdot \theta' = 0$$

$$\frac{d\theta'}{\theta'} + \frac{Pr}{2} \cdot f \cdot d\eta = 0$$

$$\theta = C_1 \int_0^{\eta} \left[\exp\left(-\frac{Pr}{2} \int_0^{\eta} f \, d\eta \right) \right] d\eta + C_2$$

Using our b.c.'s $\Rightarrow$ $\theta = 0$ at $\eta = 0 \Rightarrow C_2 = 0$

$\theta = 1$ at $\eta \to \infty \Rightarrow C_1 = \dfrac{1}{\int_0^{\infty} \left[\exp\left(-\frac{Pr}{2} \int_0^{\eta} f d\eta \right) \right] d\eta}$

So our cumbersome solution is: (3)

$$\boxed{\theta(\eta) = \frac{\int_0^{\eta} \left[\exp\left(-\frac{Pr}{2} \int_0^{\eta} f \, d\eta \right) \right] d\eta}{\int_0^{\infty} \left[\exp\left(-\frac{Pr}{2} \int_0^{\eta} f \, d\eta \right) \right] d\eta}}$$

$\Rightarrow f(\eta)$ is tabulated from the Blazius solution, so we can solve this numerically.

However, there is an easier way to solve this:
Let:

$$f(\eta^*) = Pr^{2/3} \cdot f(\eta) \quad (4) \qquad \text{and} \qquad \eta^* = \eta \cdot Pr^{1/3} \quad (5)$$

Back substitute (4) and (5) into (2)

$$\theta'' + \frac{1}{2} f(\eta^*) Pr^{1/3} \frac{\partial \theta}{\partial \eta} = 0$$

From our definition (eq. 5) $\Rightarrow$ $\eta^* = \eta Pr^{1/3}$
$d\eta^* = Pr^{1/3} d\eta$

$$\theta'' + \frac{1}{2} f(\eta^*) Pr^{2/3} \underbrace{\frac{\partial \theta}{\partial \eta Pr^{1/3}}}_{\partial \eta^*} = 0 \quad (6)$$

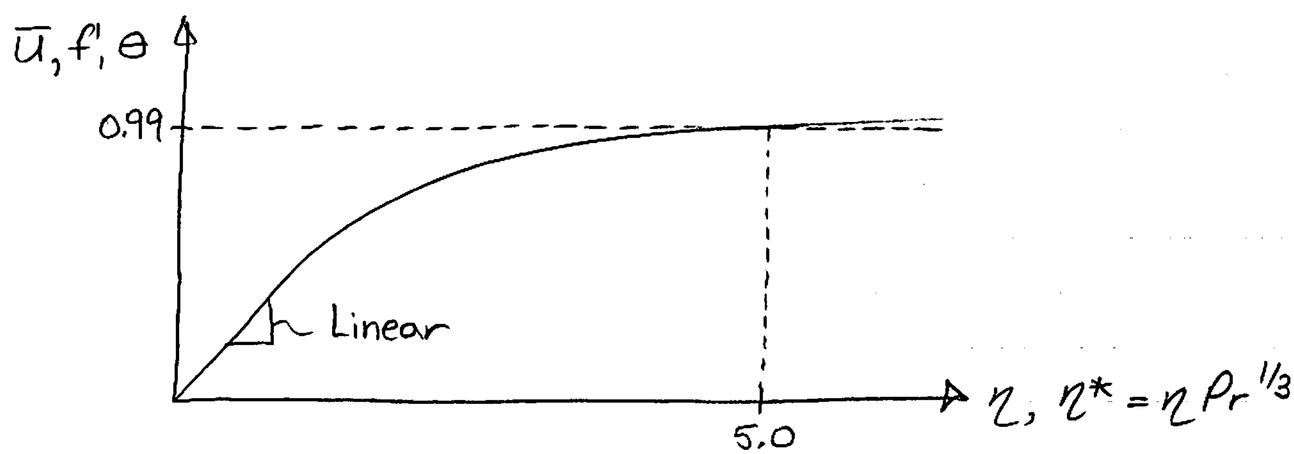
Multiply (6) through by $Pr^{-2/3}$

$$\underbrace{\frac{1}{Pr^{2/3}} \Big(\frac{\partial^2 \theta}{\partial \eta^2}}_{\partial \eta^{*2}} + \frac{1}{2} f(\eta^*) Pr^{2/3} \frac{\partial \theta}{\partial \eta^*} = 0 \Big)$$

Now our PDE becomes identical to the Blasius ODE. We already have the solution.

$$\theta'' + \frac{1}{2} f(\eta^*) \theta' = 0 \quad \Rightarrow \quad \theta(\eta^*) \Leftrightarrow \bar{u}(\eta) = f'$$

Our b.c.'s are: $\left.\begin{aligned} \theta(\eta^* = 0) &= 0 \\ \theta(\eta^* \to \infty) &= 1 \end{aligned}\right\} \; \theta = \dfrac{T - T_0}{T_\infty - T_0}$



η = hydrodynamic b.l.
η^* = thermal b.l.

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For the hydrodynamic boundary layer, we had:

$$\delta\sqrt{\frac{U_\infty}{x\nu}} = \eta(y=\delta) = 5.0 \Rightarrow \boxed{\delta(x) = \frac{5x}{\sqrt{Re_x}}}$$

Now for the thermal boundary layer, our solution is similar

$$Pr^{1/3}\,\delta_T\sqrt{\frac{U_\infty}{x\nu}} = \eta^*(y=\delta_T) = 5.0 \Rightarrow \boxed{\delta_T(x) = \frac{5x}{Re_x^{1/2}\,Pr^{1/3}}}$$

Taking the ratio of our two boundary layer thicknesses:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}} \Rightarrow \text{Kind of intuitive since } Pr = \frac{\nu}{\alpha} = \frac{\text{hydrdynamics}}{\text{energy}}$$

Note before on page 45 of notes, we found with scaling that:

$$\delta_T \sim L\,Pr^{-1/3}\,Re_L^{-1/2} \quad \Big\} \text{ Here we have } \delta_T(L) = 5L\,Re_L^{1/2}\,Pr^{1/3}$$

Now we can solve for our heat transfer (note we know: $h \sim \frac{k}{\delta_T}$)

$$q''\Big|_{y=0} = -k\frac{\partial T}{\partial y}\Big|_{y=0} \Rightarrow \theta = \frac{T-T_0}{T_\infty - T_0} \quad ; \quad \partial\theta = \frac{\partial T}{T_\infty - T_0}$$

$$\eta^* = \eta\,Pr^{1/3} = y\left(\frac{U_\infty}{x\nu}\right)^{1/2} Pr^{1/3}$$

$$\partial\eta^* = \partial y\left(\frac{U_\infty}{x\nu}\right)^{1/2} Pr^{1/3}$$

$$q''\Big|_{y=0} = -k\,(T_\infty - T_0)\left(\frac{U_\infty}{x\nu}\right)^{1/2} Pr^{1/3}\,\frac{\partial\theta}{\partial\eta^*}\Big|_{\eta^*=0}$$

$$= +\frac{k\,(T_0 - T_\infty)}{x}\left(\frac{U_\infty x}{\nu}\right)^{1/2} Pr^{1/3}\,f''(0)$$

Remember we already know $f \Rightarrow f = \frac{a_2\eta^2}{2!} - \frac{a_2\eta^5}{2\cdot 5!} + \frac{11}{4}\frac{a_3^3\eta^8}{8!}\ldots$
From page 50 of notes

$$f''(0) = a_2 = 0.332$$

$$q''\big|_{y=0} = \frac{k\Delta T}{x} Re_x^{1/2} Pr^{1/3} a_2$$

$$Nu_x = \frac{hx}{k} = \underbrace{\frac{q''|_{y=0}}{\Delta T}}_{h} \cdot \frac{x}{k} = a_2 Re^{1/2} Pr^{1/3}$$

$$\boxed{Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}}$$

⇒ From scaling: $Nu_L \sim Re_L^{1/2} Pr^{1/3}$
⇒ Valid for $Pr > 0.5$, $T_o = \text{constant}$

Colburn Analogy

We can relate our fluid flow & heat transfer solutions

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho u_\infty^2} = \frac{2a_2}{Re_x^{1/2}} \quad \text{(Skin friction coefficient)}$$

$$Nu_x = a_2 Re_x^{1/2} Pr^{1/3}$$

Let's try the following

$$\frac{Nu_x}{Re_x \cdot Pr} = \frac{a_2 \cancel{Re_x}^{1/2} Pr^{1/3}}{Re_x \cdot Pr} = \frac{a_2}{Re_x^{1/2} Pr^{2/3}} = \frac{1}{2}\overbrace{\left(\frac{2a_2}{Re_x^{1/2}}\right)}^{C_{f,x}}\frac{1}{Pr^{2/3}}$$

So whats the point of dividing by Pr?
Let's expand our definition:

$$\frac{Nu_x}{Re_x \cdot Pr} = \frac{hx}{k}\cdot\frac{\mu}{\rho u_\infty x}\cdot\frac{\alpha}{\nu} = \frac{h}{\cancel{k}}\cdot\frac{\cancel{\mu}}{\cancel{\rho} u_\infty}\cdot\overbrace{\frac{\cancel{k}}{\rho C_p}}^{\alpha}\cdot\overbrace{\frac{\cancel{\rho}}{\cancel{\mu}}}^{\nu} = \frac{h}{\rho C_p u_\infty}$$

$$\boxed{\frac{Nu_x}{Re_x \cdot Pr} = \frac{h}{\rho C_p u_\infty} = St \equiv \text{Stanton Number}}$$

$$\boxed{St = \frac{\text{heat transfer to fluid}}{\text{thermal capacity of fluid}} \sim \frac{T_w - T_b}{T_{b,t=2} - T_{b,t=1}}}$$

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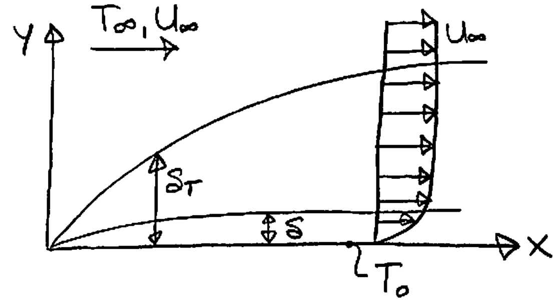
We can now say the following:

$$\boxed{St \cdot Pr^{2/3} = \frac{C_{f,x}}{2} = j_H \equiv \text{Colburn j-factor}} \quad \text{(or Colburn-Chilton analogy)}$$

Very useful for heat transfer analysis as it relates transport properties to one another, (heat, momentum, and mass transfer).

Note the previous solution is valid for $Pr \approx 1$, or $Pr >> 1$.
What if $Pr \ll 1$.



Think about why this changes:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

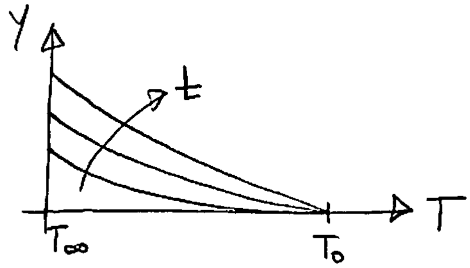
If $Pr \ll 1$, $u \not\propto Re_x^{-1/2}$ since $\delta \ll \delta_T$
Hence $u \sim U_\infty$. Our energy equation becomes:

$$U_\infty\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

From continuity: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow u = U_\infty = \text{constant} \Rightarrow v = \text{const.} = 0$ (outside the b.l.)

$$U_\infty\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2} \Rightarrow \text{but} \quad x = U_\infty \cdot t \Rightarrow dx = U_\infty dt$$

$$\cancel{U_\infty}\frac{\partial T}{\cancel{U_\infty} dt} = \alpha\frac{\partial^2 T}{\partial y^2} \Rightarrow \boxed{\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial y^2}} \Rightarrow \text{Transient conduction!!!}$$



$$q''_{y=0} = \frac{k\Delta T}{(\pi\alpha t)^{1/2}} \Rightarrow \text{Solved in ME420, Look up in conduction}$$

$$Nu_x = \left(\frac{q''_{y=0}}{\Delta T}\cdot\frac{x}{k}\right) = \frac{x}{(\pi\alpha t)^{1/2}}; \quad t = \frac{x}{U_\infty}$$

$$\boxed{Nu_x = \frac{1}{\pi^{1/2}}\cdot Re_x^{1/2} Pr^{1/2}} \Rightarrow Pr < 0.5, \; T_o = \text{constant}$$