Average Quantities

$$\underbrace{\bar{h} = \frac{\overline{q''}}{\Delta T}}_{T_o = \text{constant}} \quad \text{or} \quad \underbrace{\bar{h} = \frac{q''}{\overline{\Delta T}}}_{q''|_{y=0} = \text{constant}}$$

For T_o = constant:

$$\bar{h} = \frac{1}{\Delta T}\left[\frac{1}{L}\int_0^L q'' dx\right] = \frac{1}{L}\int_0^L h\, dx$$

For $q''|_{y=0}$ = constant:

$$\bar{h} = \frac{q''|_{y=0}}{\frac{1}{L}\int_0^L \Delta T\, dx}$$

The average $Nu_L = \overline{Nu_L} = \frac{\bar{h}L}{k} \neq \frac{1}{L}\int_0^L Nu_x dx$

For a flat plate:

$$\bar{h} = \frac{1}{L}\int_0^L h\, dx \Rightarrow h = \frac{k}{x} Nu_x$$

$$\bar{h} = \frac{0.332 k Pr^{1/3}}{L}\sqrt{\frac{U_\infty}{\nu}}\int_0^L \frac{x^{1/2}}{x} dx \Rightarrow \boxed{\bar{h} = 0.664\, Re_L^{1/2}\, Pr^{1/3} \cdot \frac{k}{L}}$$

↳ T_o = constant, $Pr > 0.5$

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k} = 0.664\, Re_L^{1/2} Pr^{1/3}} \quad Pr > 0.5$$

$$\boxed{Nu_L = 1.13\, Re_L^{1/2} Pr^{1/2}} \quad Pr < 0.5$$

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Some Observations and Notes

So far, our results are valid for the following conditions

1) $Re_x = \frac{\rho U_\infty x}{\mu}$ or $Re_L = \frac{\rho U_\infty L}{\mu} < 5.0 \times 10^5$ (Laminar)

2) $Ma = \frac{U_\infty}{\text{sound speed}} < 0.3$ (Incompressible)

3) $Ec \equiv$ Eckert Number $= \frac{U_\infty^2}{C_p(T_0 - T_\infty)} \ll 1 \Rightarrow$ (Viscous dissipation heating is negligib)

4) Evaluate fluid properties at the b.l. film temp.:

$$T_f = \frac{T_0 + T_\infty}{2}$$

*5) $h_x \sim \frac{1}{x^{1/2}} \Rightarrow$ As $x \to 0$, $h_x \to \infty$

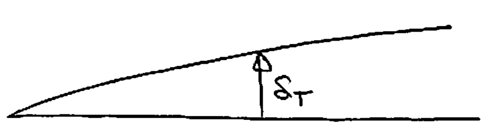
The boundary layer model breaks down in the region of $x=0$.

6) So far, we've only dealt with T_0 = constant. We will solve for the $q''_{y=0}$ = constant case later, with the integral technique.

* Note although the b.l. solution diverges at $x \to 0$, in real life, h_x is actualy higher in this region. Hence its more benifitial to re-start your b.l. as often as possible to minimize δ_T and maximize entrance effects. We will discuss these later in the class.

i.e.

$\Rightarrow$ Good, $h\uparrow$, however $\tau\uparrow$ as well



$\Rightarrow$ Bad, $h\downarrow$, however $\tau\downarrow$ as well

Similarity Solutions for Flow with Longitudinal Pressure Grad.

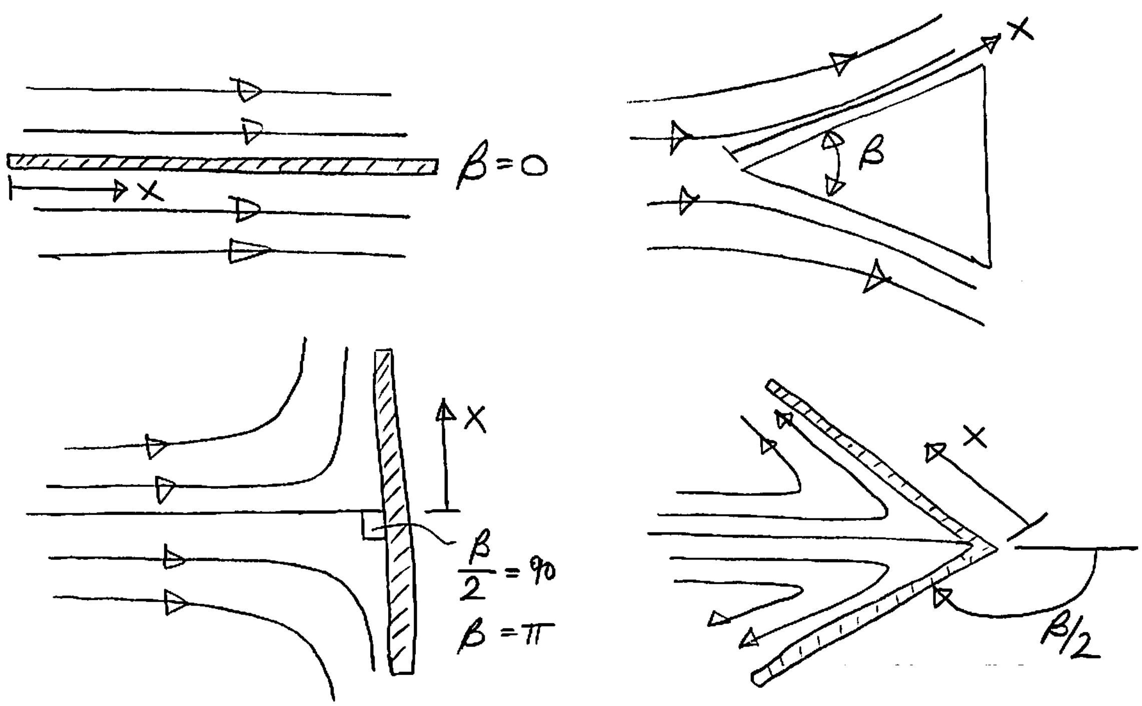
So how do we deal with flows which have longitudinal pressure gradients?

It turns out there is a class of similarity solutions that work for potential flow problems, in 2D.

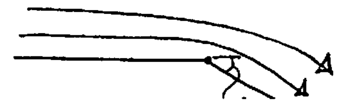
For 2D potential flows:

$$U_\infty(x) = C x^m \Rightarrow$$ For a derrivation of this, visit a potential flow fluids textbook.

$$m = \frac{\beta}{2\pi - \beta} = \frac{x}{U_\infty} \cdot \frac{dU_\infty}{dx}$$



Note for these cases, we cannot assume $\frac{\partial P}{\partial x} = 0$, except for $\beta = 0$. The varying cross section in each flow induces a pressure change in the longitudinal direction.

Also note, if $\beta < 0$, this means: 

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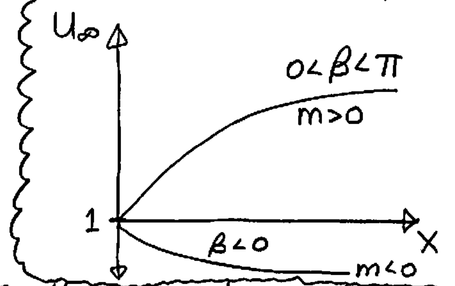
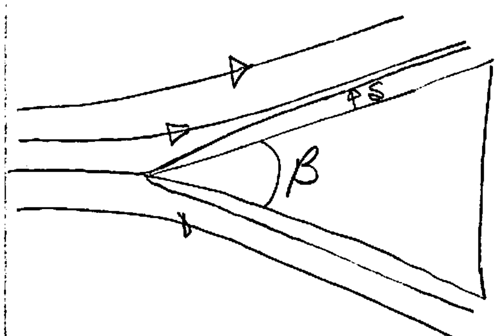
Aside: What $U_\infty = x^m$ looks like:



Our b.l. equation (momentum) becomes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

Taking a streamline at the edge of the boundary layer, or within a b.l. reveals: (note $\partial P/\partial y \ll \partial P/\partial x$ still)



$$P_\infty + \frac{1}{2}\rho U_\infty^2 = \text{constant}$$

$$\frac{\partial P_\infty}{\partial x} + \rho U_\infty \frac{\partial U_\infty}{\partial x} = 0$$

$$-\frac{\partial P}{\partial x} = \rho U_\infty \frac{\partial U_\infty}{\partial x} \quad (2)$$

Back substituting (2) into (1), we obtain:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty\frac{\partial U_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (3)$$

We know $U_\infty = Cx^m \Rightarrow U_\infty \frac{\partial U_\infty}{\partial x} = Cx^m mCx^{m-1}$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{U_\infty^2 m}{x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (4)$$

To solve equation (4) we can actually use the identical similarity variable we defined earlier.

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

Instead of using the same procedure as before (i.e. defining $f' = u/U_\infty$, & solving for $u, v = f(f', \eta)$. We can define a streamfunction $\Psi(x,y)$:

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}$$

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To check if this works, we can check continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0 \Rightarrow$$ Satisfies continuity (Note, only valid for analytic functions)

$$\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x} = 0$$

Now we can do one more thing:

$$f' = \frac{u}{U_\infty} \Rightarrow u = \frac{\partial \psi}{\partial y}$$

$$f' = \frac{1}{U_\infty}\cdot\frac{\partial \psi}{\partial y}$$

$$U_\infty f' = \frac{\partial \psi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} \Rightarrow \eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}$$

$$U_\infty^{1/2} f' \frac{\sqrt{\nu x}}{\cancel{\sqrt{U_\infty}}} = \frac{\partial \psi}{\partial \eta}$$

$$\psi = \sqrt{U_\infty \nu x}\int f' d\eta \Rightarrow \boxed{\psi = (U_\infty \nu x)^{1/2} f} \quad (5)$$

Now we can transform our b.l. equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{U_\infty^2 m}{x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (6)$$

Back substitute (5) and $u = \frac{\partial \psi}{\partial y}$, $\underbrace{v = -\int \frac{\partial u}{\partial x} dy}_{\text{Continuity}}$ into (6)

Also, you will need:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right)$$ $U_\infty = f(x)$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial \psi}{\partial \eta} = (U_\infty \nu x)^{1/2} f'$$; Remember $U_\infty = Cx^m$

Simplifying, we obtain: $$\boxed{f''' + \frac{1}{2}(m+1)ff'' + m(1-(f')^2) = 0}$$

↳ For full derrivation, see pg. 64A - 64D

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<u>Extra Derivation:</u> Falkner - Skan Momentum Equation

We want to go from PDE to ODE:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{U_\infty^2 m}{x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

Let's assume our streamfunction formulation:

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = \frac{\partial \psi}{\partial x} \; ; \; \eta = y\sqrt{\frac{U_\infty}{\nu x}} \; ; \; \psi = (U_\infty \nu x)^{1/2} f \quad (2)$$

We will need the following quantities to help us:

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) = \sqrt{\frac{U_\infty}{\nu x}} \Rightarrow \boxed{\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}} \quad (3)$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{U_\infty}{\nu x}}\right) \Rightarrow U_\infty = x^m$$

$$= \frac{\partial}{\partial x}\left(y\sqrt{\frac{x^m}{\nu x}}\right) = \frac{y}{\sqrt{\nu}}\frac{\partial}{\partial x}\sqrt{x^{m-1}}$$

$$= \frac{y}{\sqrt{\nu}}\,\frac{1}{2}(m-1)\,x^{\frac{1}{2}(m-1)-1}$$

$$= \frac{y}{\sqrt{\nu}}\,\frac{1}{2}\,\frac{m-1}{x}\sqrt{\frac{x^m}{x}}$$

$$= y\,\frac{(m-1)}{2x}\sqrt{\frac{U_\infty}{\nu x}}$$

$$\boxed{\frac{\partial \eta}{\partial x} = \frac{(m-1)}{2x}\eta} \quad (4)$$

Also remember that $\frac{u}{U_\infty} = f' \Rightarrow \boxed{u = U_\infty f'}$ (5)

And from mass conservation, we can solve for v:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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We know that:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(U_\infty f'\right) = \frac{\partial U_\infty}{\partial x} f' + \frac{\partial f'}{\partial x} U_\infty$$

$$= \frac{\partial}{\partial x}\left(x^m\right) f' + \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \cdot U_\infty$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \underbrace{m x^{m-1}}_{m U_\infty x^{-1} \text{ since } U_\infty = x^m} f' + U_\infty f'' \frac{(m-1)}{2x} \eta} \quad ⑥$$

So now we can solve for v:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \text{(from continuity)}$$

$$\frac{\partial v}{\partial y} = -\frac{U_\infty}{x}\left(m f' + (m-1) f'' \eta \cdot \frac{1}{2}\right)$$

$$v = \int -\frac{U_\infty}{x}\left(m f' + \frac{(m-1)}{2} f'' \eta\right) dy$$

We know $\eta = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow dy = d\eta \cdot \sqrt{\frac{\nu x}{U_\infty}}$

$$v = -\frac{U_\infty m}{x}\sqrt{\frac{\nu x}{U_\infty}} \int f' d\eta - \frac{U_\infty (m-1)}{2x}\sqrt{\frac{\nu x}{U_\infty}} \underbrace{\int f'' \eta \, d\eta}_{\text{We've solved this before using IBP. Look on pg.}}$$

$$v = -\frac{U_\infty m}{x}\sqrt{\frac{\nu x}{U_\infty}} f - \frac{U_\infty (m-1)}{2x}\sqrt{\frac{\nu x}{U_\infty}} \left(\eta f' - f'\right)$$

Expanding this:

$$v = -\frac{U_\infty m}{x}\sqrt{\frac{\nu x}{U_\infty}} f - \frac{U_\infty m}{2x}\sqrt{\frac{\nu x}{U_\infty}} \left(\eta f' - f\right) + \frac{U_\infty}{2x}\sqrt{\frac{\nu x}{U_\infty}} \left(\eta f' - f\right)$$

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One more step of expansion:

$$V = -\frac{U_\infty}{x}\sqrt{\frac{\upsilon x}{U_\infty}} \cdot \left[mf + \frac{m\eta}{2}f' - \frac{m\cancel{f}}{\cancel{2}} - \frac{\eta f'}{2} + \frac{f}{2}\right]$$

$$\boxed{V = -\frac{U_\infty}{x}\sqrt{\frac{\upsilon x}{U_\infty}} \cdot \left[\frac{mf}{2} + \frac{m\eta}{2}f' - \frac{\eta f'}{2} + \frac{f}{2}\right]} \quad (7)$$

Now it becomes easy. We just have to plug & chug! Let's solve the viscous term (Right hand side)

$$\upsilon \frac{\partial^2 u}{\partial y^2} = \upsilon \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(u)\right) \quad \Rightarrow u = U_\infty f'$$

$$= \upsilon \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta}(U_\infty f')\frac{\partial \eta}{\partial y}\right)$$

$$= \upsilon \frac{\partial^2}{\partial \eta^2}(U_\infty f')\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$= \cancel{\upsilon} U_\infty f''' \left(\frac{U_\infty}{\cancel{\upsilon} x}\right)$$

$$\therefore \boxed{\upsilon \frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2 f'''}{x}} \quad (8)$$

Let's do the first inertial term:

$$u\frac{\partial u}{\partial x} \Rightarrow u = U_\infty f', \quad \frac{\partial u}{\partial x} \Rightarrow \text{eqn. } (6)$$

$$= U_\infty f'\left(\frac{mU_\infty}{x}f' + U_\infty f''\frac{(m-1)}{2x}\eta\right)$$

$$\boxed{u\frac{\partial u}{\partial x} = \frac{U_\infty^2}{x}\left(m(f')^2 + \frac{m}{2}\eta f' f'' - \frac{1}{2}\eta f' f''\right)} \quad (9)$$

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Now for the second inertial term: Eqn. ③

$$v\frac{\partial u}{\partial y} \Rightarrow v \Rightarrow \text{eqn. } ⑦, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial(U_\infty f')}{\partial \eta}\sqrt{\frac{U_\infty}{\nu x}}$$

$$= U_\infty f''\sqrt{\frac{U_\infty}{\nu x}}$$

$$v\frac{\partial u}{\partial y} = -\frac{U_\infty}{2x}\sqrt{\frac{\cancel{\nu x}}{\cancel{U_\infty}}}\left[mf + m\eta f' - \eta f' + f\right]\cdot U_\infty f''\sqrt{\frac{\cancel{U_\infty}}{\cancel{\nu x}}}$$

$$\boxed{v\frac{\partial u}{\partial y} = \frac{U_\infty^2}{2x}\left(\eta f'f'' - mff'' - m\eta f'f'' - ff''\right)} \quad ⑩$$

Putting ⑧, ⑨, & ⑩ together: ⑨ + ⑩ = ⑧ + $\frac{U_\infty^2 m}{x}$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{U_\infty^2 m}{x}$$

$$\frac{U_\infty^{\cancel{2}}}{\cancel{x}}\Big(m(f')^2 + \cancel{\frac{m}{2}\eta f'f''} - \cancel{\frac{1}{2}\eta f'f''} + \cancel{\frac{1}{2}\eta f f''} - \frac{m}{2}ff''$$
$$- \cancel{\frac{1}{2}m\eta f'f''} - \frac{1}{2}ff''\Big) = \frac{U_\infty^{\cancel{2}}}{\cancel{x}}\left(m + f'''\right)$$

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

$$\boxed{f''' + \frac{1}{2}(m+1)ff'' + m\left(1-(f')^2\right) = 0}$$

↳ Falkner-Skan Wedge Flow Momentum Equation.

Now you try the energy eqn!

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Note our boundary conditions remain the same:

$$f(0) = 0 \;;\; f'(0) = 0 \;;\; f'(\infty) = 1$$

Note, the solution can be found numerically and is usually tabulated.

Typically for flow problems we need to solve for shear (τ)

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \Rightarrow u = f' U_\infty$$

$$= \mu U_\infty \frac{\partial f'}{\partial y}\bigg|_{y=0} = \mu U_\infty \left[\frac{\partial f'}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial y}}_{\sqrt{\frac{U_\infty}{\nu x}}}\right]_{y=0}$$

Simplifying: (see page 51 for a similar simplification)

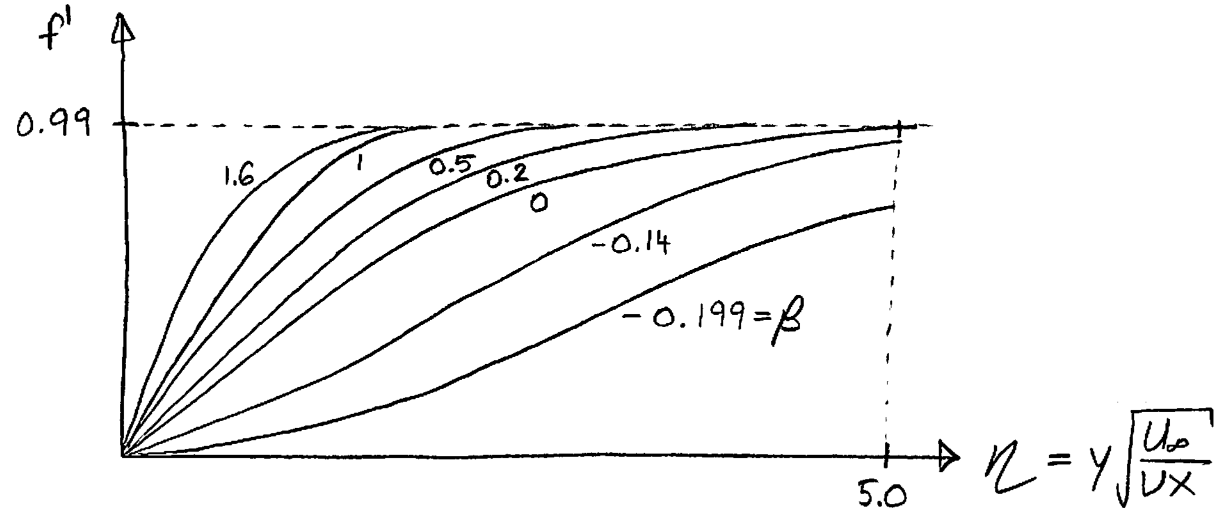
$$\boxed{C_{f,x} = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} = \frac{2f''(0)}{Re_x^{1/2}}} \Rightarrow \text{Note, } \boxed{Re_x = \frac{C x^{m+1}}{\nu}}$$

↳ Don't forget this!

Our tabulated solutions are:

β	m	$f''(0) = \frac{1}{2} C_{f,x} Re_x^{1/2}$
$2\pi = 6.28$	∞	∞
$\pi = 3.14$	1	1.233
$\pi/2 = 1.57$	1/3	0.757
$\pi/5 = 0.627$	1/9	0.512
0	0	0.332 ⇒ flat plate ($\frac{\partial P_\infty}{\partial x} = 0$)
−0.14	−0.0654	0.164
−0.199	−0.0904	0 (b.l. separation)

If we plot our velocity profile results



From this, we see that for $\beta > 0$, $m > 0$

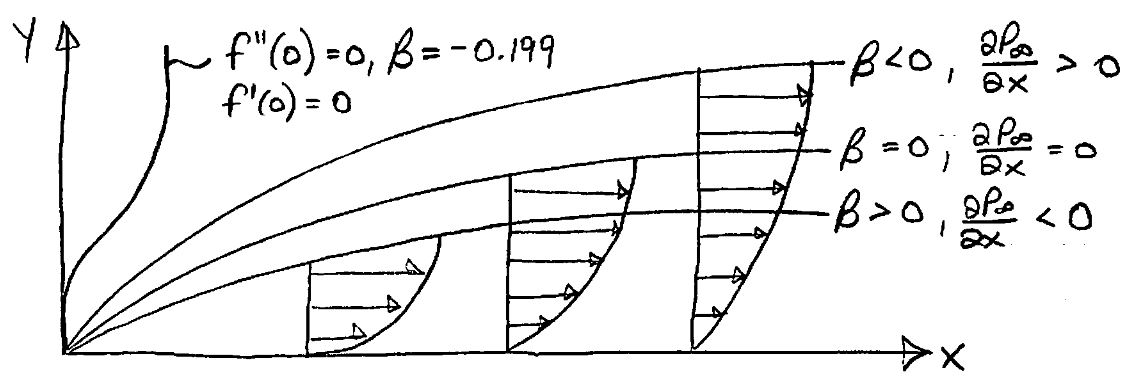
$$\frac{\partial P_\infty}{\partial x} = -\frac{\rho U_\infty^2 m}{x} < 0 \Rightarrow \text{Pressure drops as the flow accelerates}$$

This is a favourable pressure gradient, and our boundary layer gets thinner with larger x-momentum near the wall.

For $\beta < 0$, $m < 0$

$$\frac{\partial P_\infty}{\partial x} = -\frac{\rho U_\infty^2 m}{x} > 0 \Rightarrow \text{Adverse pressure gradient}$$

We see that at $\beta = -0.199$, we have $f''(0) = 0$, so this is called the b.l. separation point



Note, these solutions are called the Falkner-Skan solutions (1931)

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Heat Transfer (Falkner - Skan ⇒ Wedge Flow)
Our energy equation is exactly the same as before. We can also use the same similarity variables (page 53 of notes)

$$\theta = \frac{T - T_0}{T_\infty - T_0}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \Rightarrow \eta = y\sqrt{\frac{U_\infty}{\nu x}} \; , \; f' = \frac{u}{U_\infty}$$

Remember the following:

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left(y\sqrt{\frac{Cx^m}{\nu x}}\right)$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{Cx^m}{\nu x}}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial^2 \theta}{\partial \eta^2}\cdot\left(\frac{\partial \eta}{\partial y}\right)^2 = \frac{\partial^2 \theta}{\partial \eta^2}\cdot\frac{Cx^m}{\nu x}$$

Back substituting & doing simplification, we obtain:

$$\boxed{\theta'' + \frac{1}{2}Pr(m+1)f\theta' = 0} \quad (1)$$

This is similar to before except that Pr is replaced with $Pr(m+1)$. Our b.c.'s are identical:

$$\theta(0) = 0 \; ; \quad \theta(\infty) = 1$$

Eckert numerically integrated equation (1) & obtained (for $Pr=1$)

β	m	$Nu_x/Re_x^{1/2}$
-0.512	-0.0753	0.272
0	0	0.332
$\pi/5$	1/9	0.378
$\pi/2$	1/3	0.440
π	1	0.570

$$Nu = \frac{hx}{k}$$

⇒ Table 2.3 of Bejan [pg 63]

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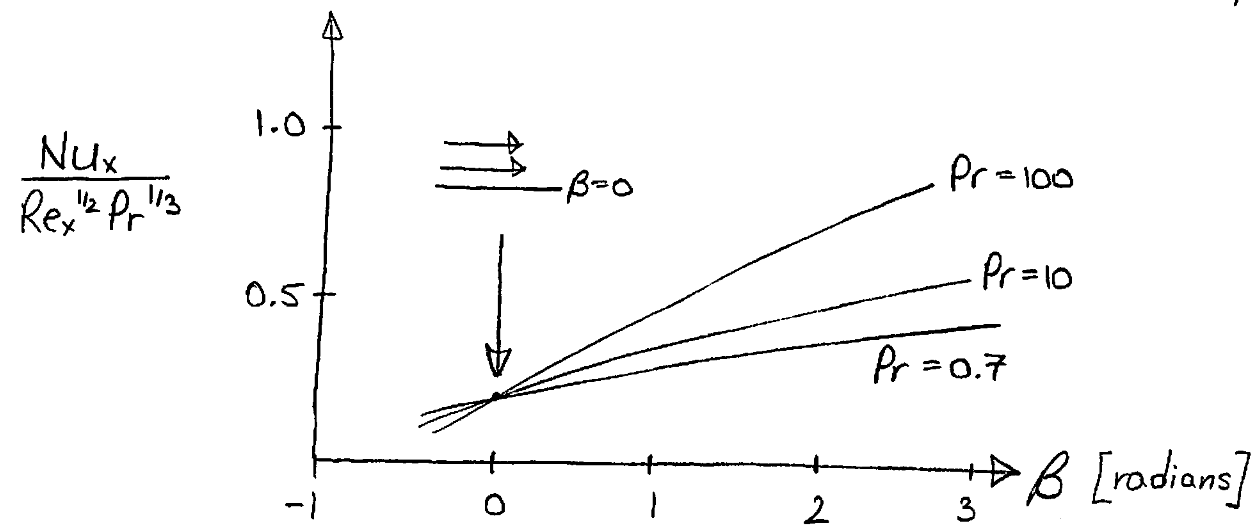
There is a good way to summarize our results graphically



For any given constant Pr, we can see that:

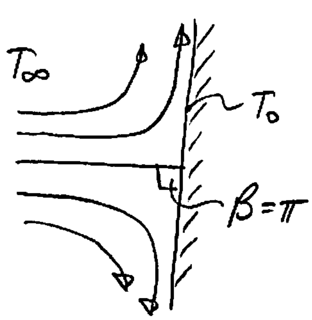
$$Nu_x Re_x^{-1/2} = \text{constant}$$

$$\frac{hx}{k}\left(\frac{xU_\infty}{\nu}\right)^{-1/2} = \text{constant} \Rightarrow h = \frac{(\text{Const.})\, k x^{-1/2} U_\infty^{1/2}}{\nu^{1/2}}$$

But for our problem, $U_\infty = Cx^m$

$$\boxed{h = \frac{(\text{const})k}{\nu^{1/2}} x^{(m-1)/2}} \Rightarrow \text{heat transfer coefficient (local)}$$

Note a spetial case here, for $m=1$, $\beta=\pi$



$$h = \frac{(\text{const})k}{\nu^{1/2}} x^{(1-1)/2} = \frac{(\text{const})k}{\nu^{1/2}} = \text{CONSTANT}$$

since h_x = constant, this implies that δ_T = const. for $m=1$. Also, δ = constant for this case.

We can say something about the average heat transfer coeff ($\bar{h}$)