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$$\bar{h} = \frac{1}{x}\int_0^x h\,dx$$

$$\bar{h} = \frac{1}{x}\int_0^x \frac{Ck}{\nu^{1/2}} x^{(m-1)/2}\,dx = \frac{2}{m+1}\frac{Ck}{\nu^{1/2}} x^{(m-1)/2}$$

$$\boxed{\frac{\bar{h}}{h} = \frac{2}{m+1}}$$ ⇒ h is the local heat transfer coefficient ($h(x)$)

Note in real experiments, for $Pr \approx 1$ ($0.5 < Pr < 10$) and $m = 1$:

$$\boxed{Nu_x = 0.57\, Re_x^{1/2} Pr^{0.4}}$$ ⇒ Jet impinging on a wall (2-D)

note: $Re_x = \frac{U_\infty x}{\nu}$

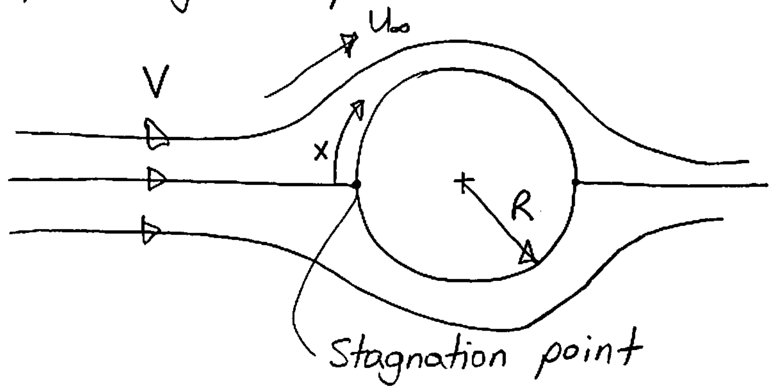
For a 3-D stagnation point:

$$\boxed{Nu_x = 0.76\, Re_x^{1/2} Pr^{0.4}}$$ ⇒ Jet impinging on a wall (3-D)

Makes sense that $h_{x,3D} > h_{x,2D}$ since we create a third dimension to dissipate energy & momentum. This is similar to turbulance!

Cylinders and Spheres

To use the above jet impingement solutions ($Nu_x Re_x^{-1/2} = \text{const}$) it is useful to know the variation of U_∞ near the stagnation point: Most solutions can be aproximated by a cylinder or sphere geometry.



$$U_\infty = \frac{2Vx}{R} \text{ for small } \frac{x}{R}$$

↳ For a cylinder

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We know our solution for a 2-D stagnation point is: $(m=1)$

$$Nu_x = 0.57\, Re_x^{1/2} Pr^{0.4}$$

$$Re_x = \frac{U_\infty x}{\nu}, \quad Nu_x = \frac{hx}{k} \quad \Rightarrow \text{Convert to } R \text{ length scale}$$

$$\frac{hx}{k} = 0.57 \left(\frac{U_\infty x}{\nu}\right)^{1/2} Pr^{0.4}$$

For a cylinder, $U_\infty = \frac{2Vx}{R}$ ($V \equiv$ far field free stream velocity)

$$\frac{h\cancel{x}}{k} = 0.57 \left(\frac{2V\cancel{x^2}}{\nu R}\right)^{1/2} Pr^{0.4}$$

Multiply both sides by R

$$\frac{hR}{k} = 0.57\,(2)^{1/2} \left(\frac{VR}{\nu}\right)^{1/2} Pr^{0.4}$$

$$\boxed{Nu_R = \frac{hR}{k} = 0.81\, Re_R^{1/2} Pr^{0.4}}$$

$0.5 < Pr < 10$, $0 < Re_R < 5\times10^5$ (Laminar)

$\Rightarrow$ For a cylinder near stagnation pt.

Note, we can do the same analysis for a sphere:

$$U_\infty = \frac{3Vx}{2R} \quad \text{for small } \frac{x}{R} \quad \Rightarrow \text{Sphere (3-D)}$$

Following the same steps as above, we obtain:

$$\boxed{Nu_R = 0.93\, Re_R^{1/2} Pr^{0.4}}$$

$0.5 < Pr < 10$, $0 < Re_R < 5\times10^5$ (Laminar)

$\hookrightarrow$ For a sphere near stagnation point.

Note, these are not equivalent to $\overline{Nu_R} = \frac{\bar{h}R}{k}$!

Note both of these solutions are for $Re_R < 5\times10^5$ which is well beyond when vortices form ($Re_R \approx 40$). However, vortices are on the trailing edge, and these apply to the leading edge.

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Blowing & Suction at the Wall (Similarity Solutions)

So far, we have always assumed that $v|_{y=0} = 0$ since our wall was impermeable.

A number of applications use suction ($v|_{y=0} < 0$) or blowing ($v|_{y=0} > 0$). For example porous surfaces or surfaces with condensation or evaporation.

Our fundamental equations don't change, mainly:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Momentum}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \Rightarrow \text{Energy}$$

$\Updownarrow$ Using our streamfunction formulation (Ψ)

$$f''' + \frac{1}{2}(m+1)ff'' + m\left(1-(f')^2\right) = 0 \Rightarrow \text{Momentum} \quad (1)$$

$$\theta'' + \frac{1}{2}Pr(m+1)f\theta' = 0 \Rightarrow \text{Energy} \quad (2)$$

Note, our boundary conditions before were: (in momentum)

Dimensional	Similarity Solution $\left(f' = \frac{u}{U_\infty}\right)$
1) $u(y=0) = 0$	1) $f'(\eta = 0) = 0$; $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$
2) $u(y\to\infty) = U_\infty$	2) $f'(\eta\to\infty) = 1$
3) $v(y=0) = 0$	3) $v = \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}\cdot(\eta f' - f)$ [From mass conservat.] Pg. (48) at $y=0$, $\eta = 0$ so $f(0) = 0$

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However, now that we have arbitrary blowing or suction, our third boundary condition must change if we want to attempt a similarity solution.

Note, for an arbitrarily perscribed $v|_{y=0}$, equations ① and ② on the previous page must be solved numerically.

For our case, we seek a similarity solution knowing:

$$v = -\frac{\partial \psi}{\partial x}, \quad \psi = \sqrt{\nu x U_\infty} \cdot f, \quad U_\infty = Cx^m$$

If we combine these, we obtain

$$v = -f \frac{m+1}{2} x^{(m-1)/2} \sqrt{C\nu} - Cy \frac{m-1}{2} f' x^{m-1}$$

But we know for our case, at $y=0$ or $\eta=0$, $v = V_s$

$$v\Big|_{\eta=0} = V_s = -f \frac{m+1}{2} x^{(m-1)/2} \sqrt{C\nu} - \underbrace{C(0) \frac{m-1}{2} f'(0) x^{m-1}}_{0}$$

We know $C = U_\infty / x^m$

$$\boxed{V_s = -\frac{m+1}{2} f(0) \frac{U_\infty}{\sqrt{U_\infty x / \nu}}}$$

or

$$\boxed{f(0) = -\frac{2}{m+1} \frac{V_s}{U_\infty} \sqrt{\frac{U_\infty x}{\nu}} = -\frac{2}{m+1} \frac{V_s}{U_\infty} Re_x^{1/2}}$$

$\Rightarrow$ Our new boundary condition.

Since we know that f is a similarity solution, and f is a function of η only, for $\eta=0$ $(y=0)$, f must be a constant for all solutions.

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So to obtain a true similarity solution, our b.c.'s become:

$$f(0) = -\frac{2}{m+1}\frac{V_s}{U_\infty} Re_x^{1/2} = \text{constant} \quad ③$$

$$f'(0) = 0 \quad ④$$

$$f(\eta \to \infty) = 1 \quad ⑤$$

Equation ③ implies that to obtain a similarity solution,

$$f(0) = \text{constant} = -\frac{2}{m+1}\frac{V_s}{U_\infty}\sqrt{\frac{U_\infty x}{\nu}} \Rightarrow U_\infty = Cx^m$$

$$= -\frac{2}{m+1}\frac{V_s}{Cx^m}\sqrt{\frac{Cx^{m+1}}{\nu}}$$

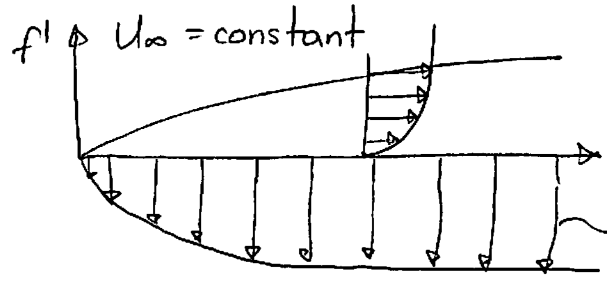
$$= -\frac{2}{m+1}\frac{\cancel{C^{1/2}}x^{1/2}\cancel{x^{m/2}}}{C\cancel{x}^{m/2}}\cdot\frac{V_s}{\sqrt{\nu}} = -\frac{2}{m+1}C^{-1/2}x^{-(m-1)/2}\frac{V_s}{\sqrt{\nu}}$$

$$\text{constant} = -\frac{2}{m+1}x^{-(m-1)/2}\cdot\frac{V_s}{\sqrt{\nu}}$$

$\boxed{V_s \sim x^{(m-1)/2}}$ for this equation to work for all x

The othe possible similarity solution is for $m=1$ (stagnation)

For $U_\infty = \text{constant}$, $m=0$, $V_s \sim \frac{1}{\sqrt{x}}$ for similarity solution:



$m=0$ case (flat horizontal plate)

$V_s \sim x^{-1/2}$

Typically, similarity solutions are characterized by:

$$\boxed{\frac{V_s}{U_\infty}Re_x^{1/2}} \equiv \text{Suction or Blowing Parameter}$$

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We use this parameter since we know in our solution that:

$$\text{Constant} = -\underbrace{\frac{2}{m+1}}_{\text{constant}} \cdot \underbrace{\frac{V_s}{U_\infty} Re_x^{1/2}}_{\text{constant}} \Rightarrow \text{from eq. } 3$$

So for any m, and suction or blowing parameter, equation 1 can be numerically solved & results tabulated.

$\frac{V_s}{U_\infty} Re_x^{1/2}$	$f''(0) = \frac{1}{2} C_{f,x} Re_x^{1/2}$	$(m=0)$
-2.5	2.59	Suction
-0.75	0.945	
-0.25	0.523	
0	0.332	Impermeable wall
$+0.25$	0.165	Blowing or Injection
$+0.375$	0.094	
$+0.5$	0.036	
$+0.619$	0	Separation

It's worth noting the order of magnitude of V_s here:

$$\frac{V_s}{U_\infty} Re_x^{1/2} \sim 1 \Rightarrow V_s \sim \frac{U_\infty}{Re_x}$$

Looking at V in our boundary layer (cons. of mass)

$$V \sim \frac{U_\infty}{Re_x^{1/2}} (\eta f' - f) \Rightarrow \text{We know } f' \,\&\, f \sim 1$$

So $V_s \sim V$ $\Rightarrow$ So for the above suction velocity results, the suction parameters are such that the suction velocity, V_s is on the same order as the transverse velocity inside the b.l., V. Note, if $\frac{V_s}{U_\infty} Re_x^{1/2} \to 0$, $V_s \ll V$ and the suction solution is not very different from the flat plate solution.

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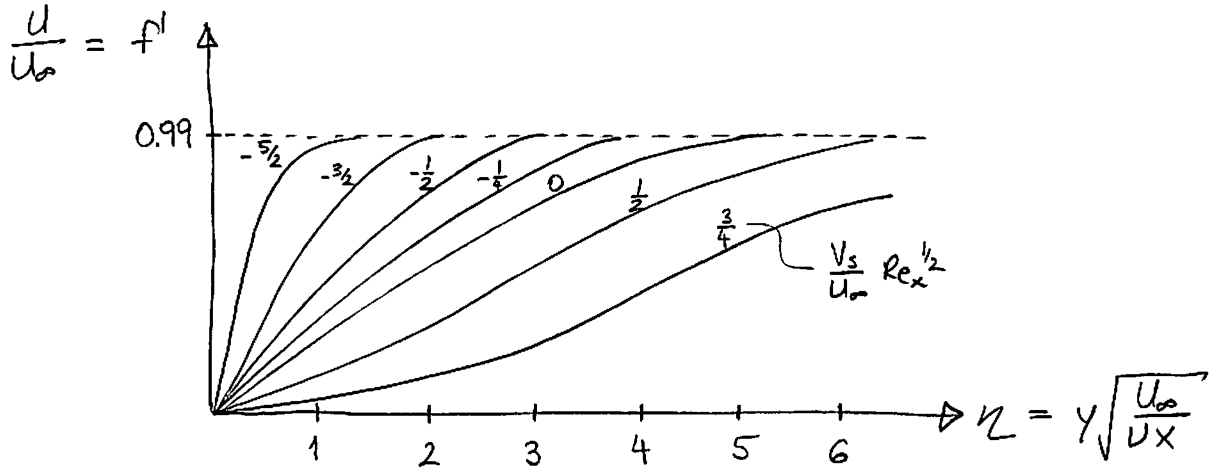
If we plot our results,



From the solutions, we see that:

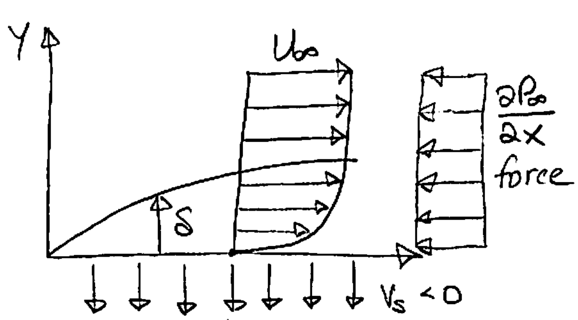
If $\frac{V_s}{U_\infty} Re_x^{1/2} < 0$ (Suction), δ decreases, τ increases

If $\frac{V_s}{U_\infty} Re_x^{1/2} > 0$ (Blowing), δ increases, τ decreases

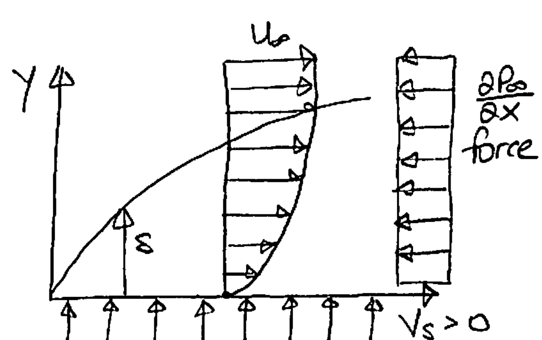
At $\frac{V_s}{U_\infty} Re_x^{1/2} = 0.619 \Rightarrow$ Separation occurs $\Rightarrow \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$

Suction tends to increase the boundary layer attachement and delay separation. This comes at the cost of increased shear and drag. Sometimes used on aircraft wings: F-16XL General Dynamic

Think of it as the suction pulls in the U_∞ stream closer to the wall, which can act against the $\partial P_\infty/\partial x$ adverse momentum close to the wall.



Suction



Blowing

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A simple particular solution can be obtained far from the leading edge.

Assuming U_∞ = constant, $m = 0$ (flat plate), $\frac{\partial P_\infty}{\partial x} = 0$

Our B.C.'s are:

$$u(y=0) = 0 \quad \text{(no slip)}$$
$$v(y=0) = -V_s \quad \text{(suction)}$$
$$u(y \to \infty) = U_\infty$$

One particular solution that satisfies these b.c.'s is where $u \neq f(x)$ or $u = f(y)$ only. Our momentum equation becomes:

$$u\cancelto{0}{\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} = -\cancelto{0}{\frac{\partial P_\infty}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

From continuity: $\cancelto{0}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow V(y) = \text{constant}$

Since $V|_{y=0} = V_s \quad \therefore \quad V(y) = V_s \quad (2)$

Substituting (2) into (1), we obtain:

$$V_s\frac{\partial u}{\partial y} + \nu\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{Solvable ODE, second order, linear}$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{V_s}{\nu}\frac{\partial u}{\partial y} = 0$$

$$u(y) = Ae^{-\lambda y} + Be^{+\lambda y}$$

$$\lambda^2 + \frac{V_s}{\nu}\lambda = 0 \Rightarrow \lambda\left(\lambda + \frac{V_s}{\nu}\right) = 0 \Rightarrow \lambda = -\frac{V_s}{\nu} \text{ or } \lambda = 0$$

$$u(y) = Ae^{+\frac{V_s}{\nu}y} + Be^{-\frac{V_s}{\nu}y} + C \Rightarrow \text{Since } u(y \to \infty) = U_\infty, \ A = 0$$
$$u(y \to \infty) = U_\infty = Be^{-\frac{V_s}{\nu}(\infty)} + C \Rightarrow C = U_\infty$$
$$u(y=0) = 0 = Be^{(0)} + U_\infty \Rightarrow B = -U_\infty$$

$$\therefore \quad \boxed{u(y) = U_\infty\left(1 - e^{-\frac{V_s y}{\nu}}\right)} \Rightarrow \text{Asymptotic solution far from the leading edge.}$$

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Note, this solution is only valid for: $\boxed{\frac{V_s}{U_\infty} Re_x^{1/2} > 2}$

Our previous tabulated solutions for $m=0$ will fall on this curve for large x.

Heat Transfer (Suction & Blowing)

Our energy equation remains the same as before:

$$\theta'' + \frac{1}{2} Pr (m+1) f \theta' = 0 \quad ; \quad \theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

B.C.'s:
$$\theta(\eta = 0) = 0$$
$$\theta(\eta \to \infty) = 1$$

To solve, we substitute our hydrodynamic solution, f, into the ODE above and numerically solve it with the same b.c.'s as before. Note, f is not unique and changes for each case of blowing & suction parameter, and m.

Looking at our case of $m=0$, $V_s \sim x^{-1/2}$

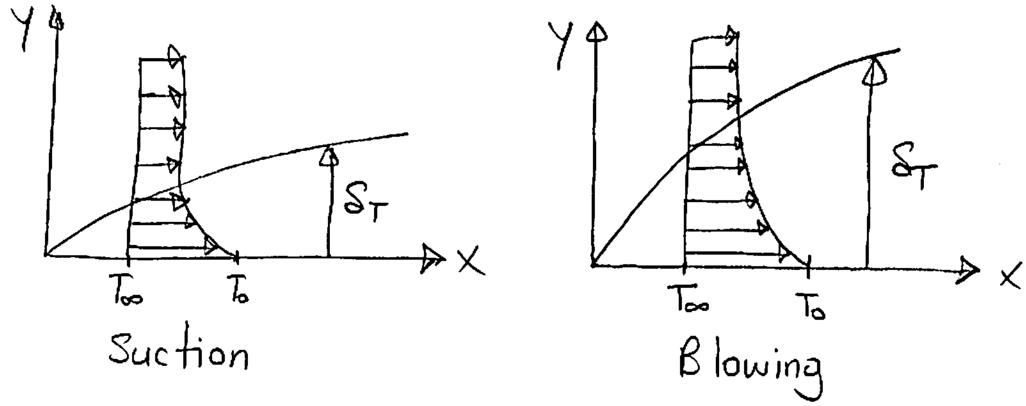
(For $m=0$)

$\frac{V_s}{U_\infty} Re_x^{1/2}$	$Nu/Re_x^{1/2}$ $Pr = 0.7$	$Nu/Re_x^{1/2}$ $Pr = 0.9$	
-2.5	1.85	2.59	Suction
-0.75	0.722	0.945	Suction
-0.25	0.429	0.523	Suction
0	0.292	0.332	Impermeable Wall
+0.25	0.166	0.165	Blowing
+0.375	0.107	0.0937	Blowing
+0.5	0.0517	0.0356	Blowing
+0.619	0	0	Separation

Note the $\frac{V_s}{U_\infty} Re_x^{1/2} = 0$ solution is the Blasius - Pohlhausen solution we've done before.

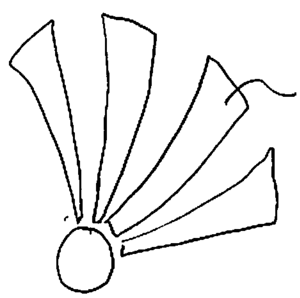
Note, $Nu_x = \frac{hx}{k} \uparrow$ for $\frac{V_s}{U_\infty} Re_x^{1/2} < 0$ (Suction, δ_T smaller)

$Nu_x = \frac{hx}{k} \downarrow$ for $\frac{V_s}{U_\infty} Re_x^{1/2} > 0$ (Blowing, δ_T larger)



Since $h \sim k \frac{\Delta T}{\delta_T} \Rightarrow h_{suction} > h_{Blowing}$

Note, this can be a means of cooling. For example



Turbine blades on a jet or a generator. Blades are colder than the hot gases ($T_0 < T_\infty$). So blowing will decrease h and cool the blade. In this case, blowing is advantageous.

Also note, these solutions are valid for the same fluid being injected or sucked. If not $\Rightarrow$ mass transfer is important!

Same as previously derived: pg. 68 & 69 of notes

$$\boxed{\frac{\bar{h}}{h} = \frac{2}{m+1}}$$