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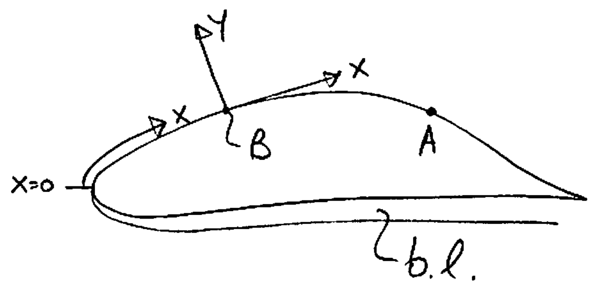
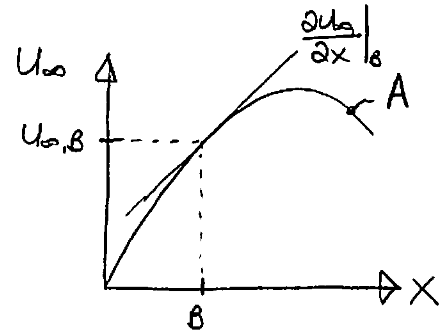
Local Similarity

Suppose we had an arbitrary shape with flow around it:



$\Leftrightarrow$



Assumption, conditions upstream have little influence on the b.l. behaviour at location X.

Thus the b.l. thickness is mainly a manifestation of local conditions.

This is only an approximate analysis. From our U_∞ profile, m can be calculated for a local wedge flow:

$$\left.\begin{aligned} U_\infty &= Cx^m \\ \frac{\partial U_\infty}{\partial x} &= Cmx^{m-1} = m\frac{U_\infty}{x} \end{aligned}\right\} \quad m = \frac{x\,\frac{\partial U_\infty}{\partial x}}{U_\infty}$$

We assume the b.l. thickness at X is identical to that of a wedge flow with:

$$\beta = \frac{2m}{m+1}$$

We can use this to solve for wall shear stress: $\tau = \mu \frac{\partial u}{\partial y}\Big|_x$

We can also solve for when separation will occur ($\beta = -0.1988$). $\quad \tau \cong \mu \frac{U_\infty}{\delta(x)}$

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Integral Methods

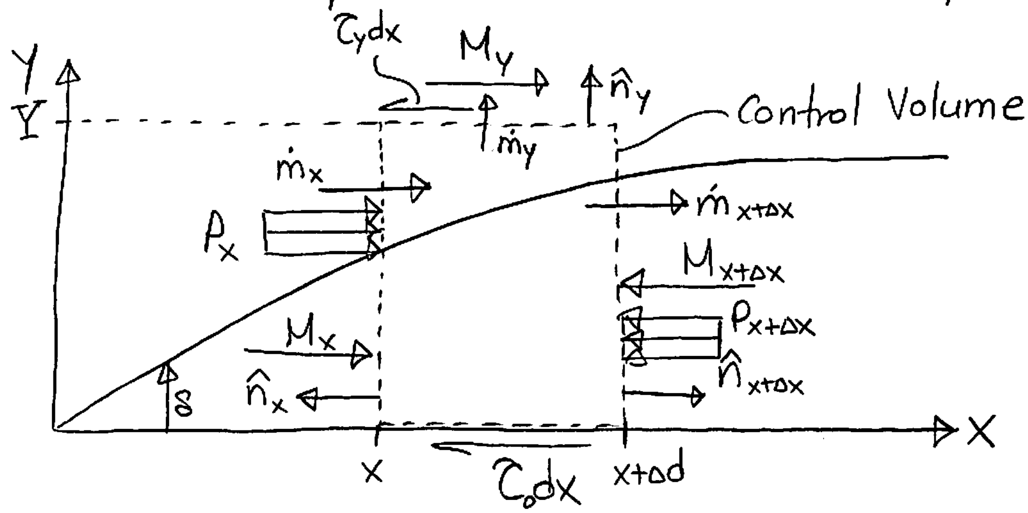
So far, we have only looked at similarity solutions, which are good for:

$\boxed{\text{Similarity}}$ — Constant T B.C.'s; Simple flows; Simple geometries

But what if $T = f(x)$? or what about $q''|_{y=0} = \text{constant}$?

For more complex flows and geometries, it turns out integral techniques are quite usefull.

Let's look at how they work. Consider a boundary layer:



Looking at each of our terms for an x-momentum balance:

$$M_x = (\dot{m}u)|_x \quad \Rightarrow \quad \dot{m}|_x = \int_0^Y \rho u\,dy$$

$$M_x = \int_0^Y \rho u^2 dy\Big|_x \quad (1)$$

$$M_{x+\Delta x} = (\dot{m}u)|_{x+\Delta x} = \int_0^Y \rho u^2 dy\Big|_{x+\Delta x} \quad (2)$$

$$M_Y = (\rho v u)|_Y \cdot \Delta x \quad (3)$$

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Now for our forces:

Taylor Series Expansion

$$(P_x - P_{x+\Delta x})Y = \left[\cancel{P_x} - \left(\cancel{P_x} + \frac{\partial P}{\partial x}\bigg|_x \Delta x + \cancel{\frac{1}{2}\frac{\partial^2 P}{\partial x^2}\Delta x^2} + \dots\right)\right] Y \quad (4)$$

$$F_{shear,0} = \tau dx\Big|_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \Delta x \quad (5)$$

$$F_{shear,Y} = \tau dx\Big|_Y = \mu \frac{\partial u}{\partial y}\bigg|_{y=Y} \Delta x \quad (6)$$

H.O.T. ≡ higher order terms

Putting everything together, we obtain:

$$\int_0^Y \rho u^2 dy\bigg|_{x+\Delta x} - \int_0^Y \rho u^2 dy\bigg|_x + (\rho v u)\Big|_Y \cdot \Delta x$$

$$= -\left(\frac{\partial P}{\partial x}\bigg|_x \Delta x + H.O.T\right)Y - \mu\frac{\partial u}{\partial y}\bigg|_0 \Delta x + \mu \frac{\partial u}{\partial y}\bigg|_Y \Delta x$$

We can Taylor expand our first two terms to obtain

$$\left[\frac{d}{dx}\int_0^Y \rho u^2 dy\bigg|_x \Delta x + H.O.T.\right] + (\rho v u)\Big|_Y \cdot \Delta x = \Sigma F$$

But note we can divide both sides by Δx, and let $\Delta x \to 0$
All of our H.O.T. will drop. We obtain:

$$\frac{\partial}{\partial x}\int_0^Y \rho u^2 dy + (\rho v u)\Big|_Y = -\frac{\partial P}{\partial x} - \mu \frac{\partial u}{\partial y}\bigg|_0 + \mu\frac{\partial u}{\partial y}\bigg|_Y \quad (7)$$

The second term above is a little tricky since we don't know v.
From mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \int_0^Y \frac{\partial v}{\partial y} = \int_0^Y -\frac{\partial u}{\partial x} \Rightarrow v|_Y - v|_0 = \int_0^Y -\frac{\partial u}{\partial x}\, dy$$

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Now we know $V_0 = 0$ (Impermeable wall)

$$v|_Y = -\int_0^Y \frac{\partial u}{\partial x}\, dy = -\frac{\partial}{\partial x}\int_0^Y u\, dy$$

Now our M_y term becomes:

$$M_y = -\rho \frac{\partial}{\partial x}\left(\int_0^Y u\,dy \cdot u|_Y\right) \Rightarrow \text{Assuming } Y > \delta,\ u|_Y = U_\infty$$

$$= -\rho \frac{\partial}{\partial x}\int_0^Y u\,dy \cdot U_\infty = -\rho \frac{\partial U_\infty}{\partial x}\int_0^Y u\,dy - U_\infty \rho \frac{\partial}{\partial x}\int_0^Y u\,dy$$

Back substituting M_y into 7, we obtain:

$$\frac{\partial}{\partial x}\int_0^Y \rho u^2\,dy - \rho\frac{\partial U_\infty}{\partial x}\int_0^Y u\,dy - U_\infty \rho \frac{\partial}{\partial x}\int_0^Y u\,dy = -\frac{\partial P}{\partial x}Y - \mu\frac{\partial u}{\partial y}\bigg|_0 + \mu\frac{\partial u}{\partial y}\bigg|_Y$$

Again, since $Y > \delta$, $\underbrace{\frac{\partial u}{\partial y}\Big|_Y = 0}$ $(u = U_\infty \neq f(y))$

$$\boxed{\frac{\partial}{\partial x}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}\frac{\partial P}{\partial x}Y + \nu\frac{\partial u}{\partial y}\bigg|_0 - \frac{\partial U_\infty}{\partial x}\int_0^Y u\,dy} \quad (8)$$

↳ Momentum Integral Boundary Layer Eqn.

Note in ME420, we automatically assumed U_∞ = constant and $\frac{\partial P_\infty}{\partial x} = 0$, so reduces to:

$$\boxed{\frac{\partial}{\partial x}\int_0^Y u(U_\infty - u)\,dy = \nu\frac{\partial u}{\partial y}\bigg|_0} \quad (9)$$

⇒ Flat Plate

⇒ U_∞ = constant, $\frac{\partial P_\infty}{\partial x} = 0$

To solve this equation, we can first non-dimensionalize

$$\frac{u}{U_\infty} = \phi(\eta)\ ; \quad \eta = \frac{y}{\delta} \Rightarrow dy = \delta\, d\eta$$

Back substituting into 9 for a flat plate and $\frac{\partial P}{\partial x} = 0$

Note also, we will integrate from 0 to $Y = \delta$ since for $y > \delta$, our integral is zero (not useful to us).

$$\frac{\partial}{\partial x}\int_0^{\delta} u(U_\infty - u)\,dy = \nu \frac{\partial u}{\partial y}\Big|_0$$

$$u = U_\infty \phi$$

$$dy = \delta\, d\eta$$

$$\frac{\partial u}{\partial y} = \frac{U_\infty \partial\phi}{\delta \partial\eta}$$

$$\frac{\partial}{\partial x}\int_0^1 U_\infty\phi\,(U_\infty - U_\infty\phi)\,\delta\, d\eta = \nu \frac{U_\infty}{\delta}\frac{\partial\phi}{\partial\eta}\Big|_0$$

$$\frac{\partial}{\partial x}\int_0^1 U_\infty^{\cancel{2}}\phi(1-\phi)\,\delta\, d\eta = \nu \frac{\cancel{U_\infty}}{\delta}\frac{\partial\phi}{\partial\eta}\Big|_0$$

Since U_∞ = constant and $\delta \neq f(y)$ only $f(x)$ we take it out

$$\delta\frac{\partial\delta}{\partial x}\int_0^1 \phi(1-\phi)\,d\eta = \frac{\nu}{U_\infty}\frac{\partial\phi}{\partial\eta}\Big|_0$$

So our equation becomes:

$$\boxed{\delta\frac{\partial\delta}{\partial x} = \frac{\nu}{U_\infty}\beta} \quad ; \quad \boxed{\beta = \frac{\phi'|_0}{\int_0^1 \phi(1-\phi)\,d\eta}} \qquad (10)$$

We know $\beta = f(y)$ and $\beta \neq f(x)$, so we can simply integrate

$$\int_0^x \delta\, d\delta = \int_0^x \frac{\nu}{U_\infty}\beta\, dx$$

$$\frac{\delta^2}{2}\Big|_0^x = \frac{\nu\beta x}{U_\infty} \Rightarrow \left(\frac{\delta}{x}\right)^2 = 2\beta \underbrace{\frac{\nu}{U_\infty x}}_{Re_x^{-1}} \Rightarrow \boxed{\frac{\delta}{x} = \sqrt{\frac{2\beta}{Re_x}}} \qquad (11)$$

We've obtained this result before with $\sqrt{2\beta} = 5.0$ (Blasius)

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The way to solve momentum integral problems is to approximate the dimensionless velocity profile $\phi(\eta)$. This method is approximate but it turns out to work very well as long as the B.C.'s are matched.

Note: It was Von-Karman (Postdoc) and Pohlhausen (student) of Prandtl who came up with this method in 1919.

Let's assume some profiles & solve for β:

$$\left.\begin{aligned}
\phi(0) &= \left.\frac{u}{U_\infty}\right|_0 = 0 \quad \text{(No Slip)} \quad (12)\\
\phi(1) &= \left.\frac{U_\infty}{U_\infty}\right|_{\eta=1} = 1 \quad \text{(Free stream)} \quad (13)\\
\phi'(1) &= \left.\frac{\partial\phi}{\partial\eta}\right|_{\eta=1} = 0 \quad \text{(no shear)} \quad (14)\\
\phi''(0) &= \left.\frac{\partial^2\phi}{\partial\eta^2}\right|_{\eta=0} = 0 \quad \text{(linear grad.)} \quad (15)
\end{aligned}\right\} \text{Boundary Conditions}$$

(15) $\Rightarrow$ Constant wall shear stress.

Assuming the following velocity profiles:

$\phi = \eta \Rightarrow$ Satisfies (12) & (13), but not (14) or (15)

$\phi = 2\eta - \eta^2 \Rightarrow$ Satisfies (12), (13) & (14), but not (15)

$\phi = \sin\left(\frac{\pi}{2}\eta\right) \Rightarrow$ Satisfies all b.c.'s

$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \Rightarrow$ Satisfies all b.c.'s

Back substituting ϕ into (10) and solving for $\sqrt{2\beta}$

$$\beta = \frac{\phi'|_0}{\int_0^1 \phi(1-\phi)\,d\eta}$$

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ϕ	$\sqrt{2\beta}$
η	3.464
$2\eta - \eta^2$	5.477
$\sin(\pi/2\,\eta)$	4.795
$\frac{3}{2}\eta - \frac{1}{2}\eta^2$	4.641

All solutions are remarkably close to $\sqrt{2\beta} = 5.0$ (Blasius exact solution!

This shows that the integral method is particularly good at approximating the correct solution from fairly crude profile assumptions.

Let's solve for shear (τ) using $\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$

$$\tau(x) = \mu \frac{\partial u}{\partial y}\bigg|_0 \quad \Rightarrow \quad \begin{aligned} u &= U_\infty \phi \\ dy &= \delta\, d\eta \end{aligned}$$

$$= \mu \frac{U_\infty \partial\phi}{\delta\, \partial\eta}\bigg|_0 \quad \Rightarrow \quad \frac{\partial\phi}{\partial\eta}\bigg|_{\eta=0} = \frac{3}{2} - \frac{3}{2}\eta^2\bigg|_0 = \frac{3}{2}$$

$$\tau(x) = \mu \frac{U_\infty}{\delta}\cdot\frac{3}{2}$$

Now if we want $C_{f,x}$

Multiply by $\left(\frac{x}{x}\right) \Rightarrow$ trick

$$C_{f,x} = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{\mu U_\infty 6}{2\delta\rho U_\infty^{\cancel{2}}} = \frac{3\nu}{\delta\rho U_\infty}\left(\frac{x}{x}\right) = \frac{3\nu}{\left(\frac{\delta}{x}\right)\rho U_\infty x}$$

But note: $Re_x = \frac{U_\infty x}{\nu} \Rightarrow C_{f,x} = \dfrac{3}{\frac{\delta}{x}\cdot Re_x}$

But we already showed $\left(\frac{\delta}{x}\right) = \left(\frac{2\beta}{Re_x}\right)^{1/2} \Rightarrow$ Back substitute

$$C_{f,x} = \frac{3}{\sqrt{2\beta}}\cdot\frac{1}{\sqrt{Re_x}} \quad \Rightarrow \text{We solved above that } \sqrt{2\beta} = 4.641$$

$$\boxed{C_{f,x} = \frac{\tau(x)}{\frac{1}{2}\rho U_\infty^2} = 0.646\, Re_x^{-1/2}}$$

$\Rightarrow$ Using similarity, we had: $C_{f,x} = 0.664\, Re_x^{1/2}$