

ME 521 - Convective Heat Transfer

Mechanisms of heat transfer

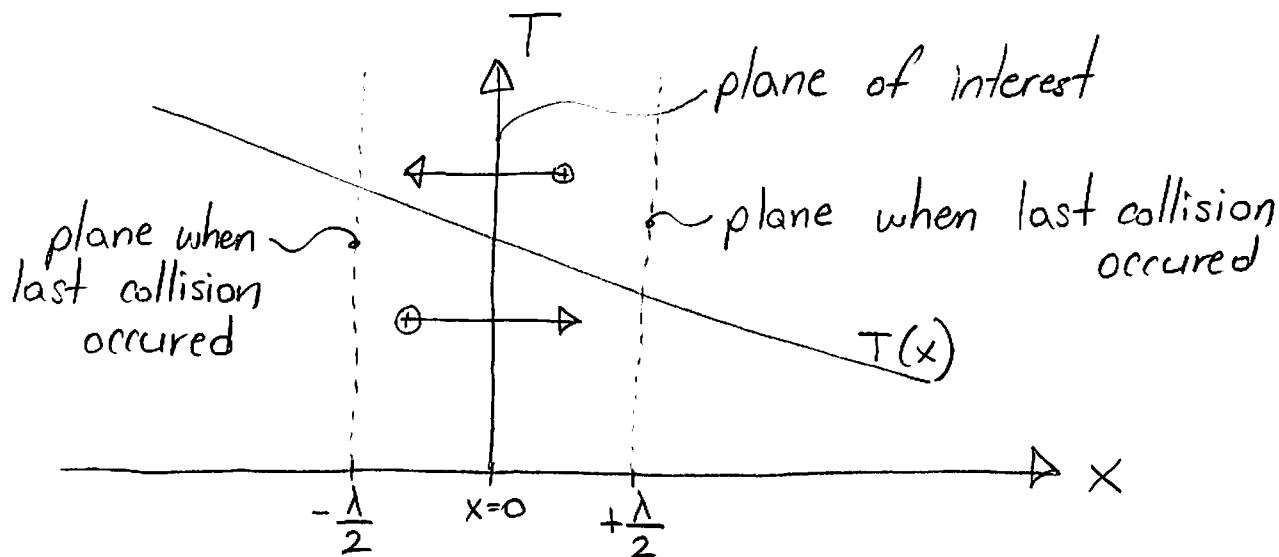
Conduction = Convection
w/ fluid motion
Radiation

So to understand convection, we need to know a little bit about what conduction is.

Conduction is a process governed by molecular diffusion. Diffusion is the process by which molecules/atoms/electrons intermingle as a result of their kinetic energy of random motion.

So wherever we have gradients, we have diffusional transport.

Let's take a look at heat:



Let's look at the energy crossing our plane at $x=0$.

$$q''_{x+} = \underbrace{nm\bar{c} \cdot C_p T(x - \frac{\lambda}{2})}_{\text{[kg/m}^3\text{] \cdot [kg/particle] [m/s]}} \left\{ \frac{\text{J}}{\text{kg} \cdot \text{K}} \right\} [\text{K}] = \left[\frac{\text{J}}{\text{kg}} \right]$$

$$\left[\frac{\text{kg}}{\text{m}^3} \right] \cdot \left[\frac{\text{kg}}{\text{particle}} \right] \left[\frac{\text{m}}{\text{s}} \right] = \left[\frac{\text{kg}}{\text{m}^2 \text{s}} \right] \quad \left[\frac{\text{kg}}{\text{m}^2} \right] \left[\frac{\text{J}}{\text{kg}} \right] = \frac{\text{J}}{\text{m}^2 \text{s}} = \boxed{W}$$
①

Just to be clear; $n = \# \text{ of molecules} / m^3 [m^{-3}]$
 $m = \text{mass per molecule} [\text{kg}]$
 $C_p = \text{specific heat capacity of molecule}$
 $\bar{c} = \text{average speed of molecule} [m/s]$

We can also say that $n \cdot m = \rho$
So now if we do the balance

$$\underbrace{\text{Energy Transfer (NET)}}_{q''_{NET,x}} = \underbrace{\text{Energy in } +x}_{q''_{x+}} - \underbrace{\text{Energy in } -x}_{q''_{x-}}$$

$$\begin{aligned} q''_{NET,x} &= \rho C_p \bar{c} \left[T\left(x - \frac{\lambda}{2}\right) - T\left(x + \frac{\lambda}{2}\right) \right] \\ &= -\rho C_p \bar{c} \left[T\left(x + \frac{\lambda}{2}\right) - T\left(x - \frac{\lambda}{2}\right) \right] \end{aligned}$$

Multiplying by λ/λ

$$q''_{NET,x} = -\rho C_p \bar{c} \lambda \underbrace{\left[\frac{T\left(x + \frac{\lambda}{2}\right) - T\left(x - \frac{\lambda}{2}\right)}{\lambda} \right]}_{\frac{\partial T}{\partial x}}$$

$$q''_{NET,x} = -\rho C_p \bar{c} \lambda \frac{\partial T}{\partial x}$$

↳ Diffusional Transport is indeed gradient dependent.

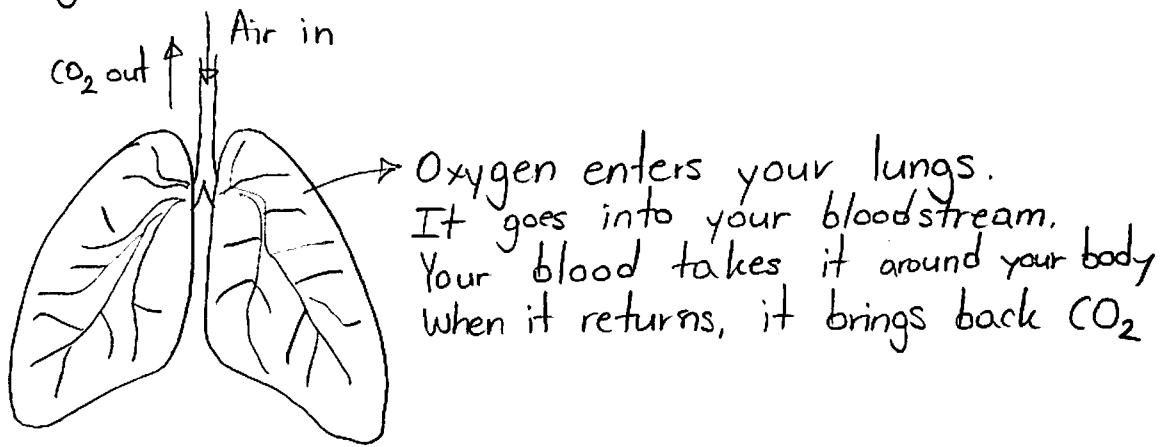
We also see that: $k = \rho C_p \bar{c} \lambda$ \Rightarrow Note, in real life $\bar{c} = f(T)$

↳ Thermal conductivity
You can actually do this same derivation with momentum to derive viscosity (μ).

So what's the big deal? Well, I won't bore you with the obvious examples of enhanced heat transfer & mass transfer. It's fairly trivial that adding bulk fluid motion to help transport the mass, momentum, energy will enhance the process.

Let's talk a little more about nature & how we breathe.

Humans & animals have lungs to help them transport air (oxygen) from the outside to their bloodstream.



As living things, our cells produce energy, & need oxygen to do so. Viruses don't do this. That is what separates us from them.

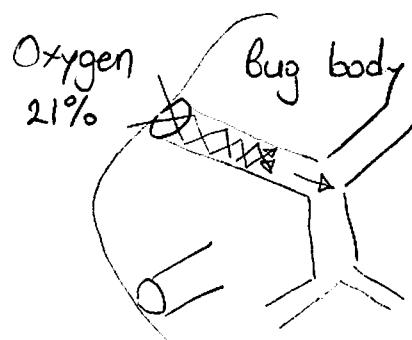
Anyways, as we breathe, we are using convection.
Note, convection = diffusion + advection

We force the fluid to come into our lungs & then interact with blood & exchange molecules.

Furthermore, our bloodstream is a very efficient convection mechanism. Blood flows through your body & to the cells that need oxygen.

So you might say "So what, this is not a biology class". Well, have you ever thought about how bugs or insects breathe?

Bugs don't have lungs or a bloodstream, so they actually rely on molecular diffusion. They have small tiny openings all over their bodies. Oxygen from the air diffuses through the openings & goes right to the cell walls where it is used.



This mass transfer mechanism is limited by the amount of diffusional resistance through the channel. Hence, bugs are usually very small (diffusion lengths are small).

Note 300 million years ago, $\text{CO}_2 = 35\%$ as compared to $\text{CO}_2 = 21\%$ today. Hence prehistoric bugs were much much bigger!

Hence convection is very important! If you didn't have it (i.e. lungs), you would be much smaller.

For more info: noticing.co/how-insects-breathe/

Fundamental Principles

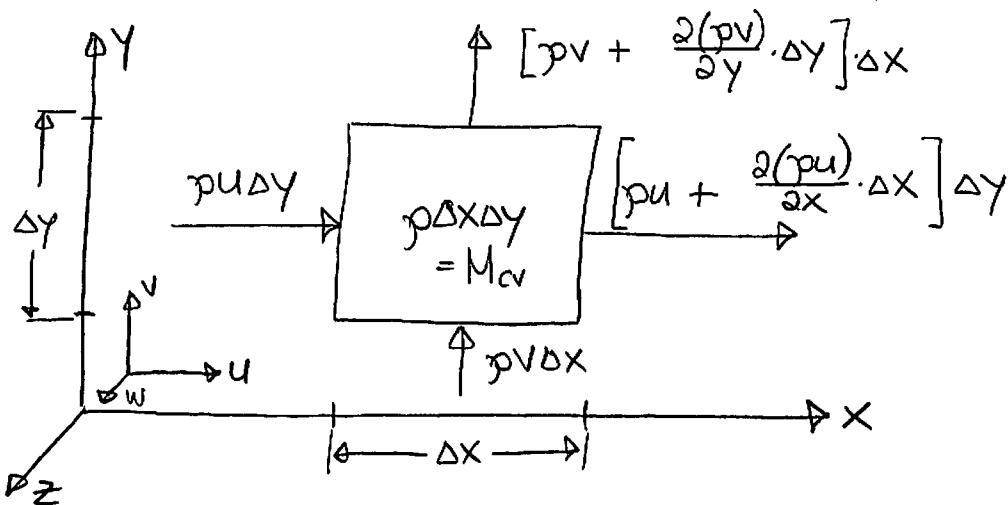
Mass Conservation

We recall from engineering thermodynamics that:

$$\frac{dM_{cv}}{dt} = \sum_{\text{inlet}} m - \sum_{\text{outlet}} m$$

M_{cv} is instantaneous mass in control volume mass flow rates out of CV

Let's draw our control volume: (in 2D)



Writing out our mass balance:

$$\frac{d}{dt} (p\Delta x \Delta y) = p u \Delta y + p v \Delta x - \left[p u + \frac{\partial(pu)}{\partial x} \cdot \Delta x \right] \Delta y - \left[p v + \frac{\partial(pv)}{\partial y} \cdot \Delta y \right] \Delta x$$

Divide through by $\Delta x \Delta y$ (constants)
& let $\Delta x \rightarrow 0, \Delta y \rightarrow 0$

I added the third dimension arbitrarily.

$$\frac{\partial p}{\partial t} + \frac{\partial(pu)}{\partial x} + \frac{\partial(pv)}{\partial y} + \frac{\partial(pw)}{\partial z} = 0$$

Expanding with the product rule:

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} + p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or

$$\boxed{\frac{\partial p}{\partial t} + p \nabla \cdot \mathbf{v} = 0} \Rightarrow \mathbf{v} \text{ is velocity vector } (u, v, w)$$

↳ Conservation of mass

$$\text{and } \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

↳ Material derivative

Note for most convection problems encountered by engineers, the density variations in the flow are much smaller than local variations in velocity.

$$\text{Thus } \Rightarrow \frac{\partial p}{\partial t} \approx 0 \text{ and: } \boxed{\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0}$$

It is important to note that the simplified formulation applies to more than just incompressible fluids. For example, gases still follow it. The better distinction is the one outlined above ($\nabla p \ll \nabla \mathbf{v}$)

For cylindrical and spherical:

$$\boxed{\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0} \Rightarrow \text{Cylindrical}$$

$$\boxed{\frac{1}{r} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} (v_\theta \sin \phi) + \frac{1}{\sin \phi} \frac{\partial v_\theta}{\partial \theta} = 0} \Rightarrow \text{Spherical}$$

Momentum Conservation (Force Balance)

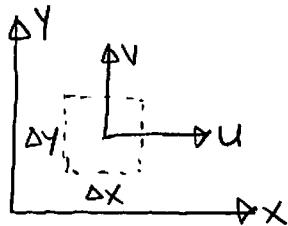
Recall from fluid mechanics (Newton's second law)

$$\underbrace{\frac{d}{dt} (M V_n)_{cv}}_{\text{mass term}} = \underbrace{\sum F_n}_{\text{Sum of forces}} + \underbrace{\sum_{\text{inlet}} m v_n - \sum_{\text{outlet}} m v_n}_{\text{Forces generated by inflow \& outflows.}}$$

where n denotes the direction of analysis, and v_n & F_n are the projected velocity & forces in the n direction.

Let's now analyze a control volume like before with a force balance in the x -direction only.

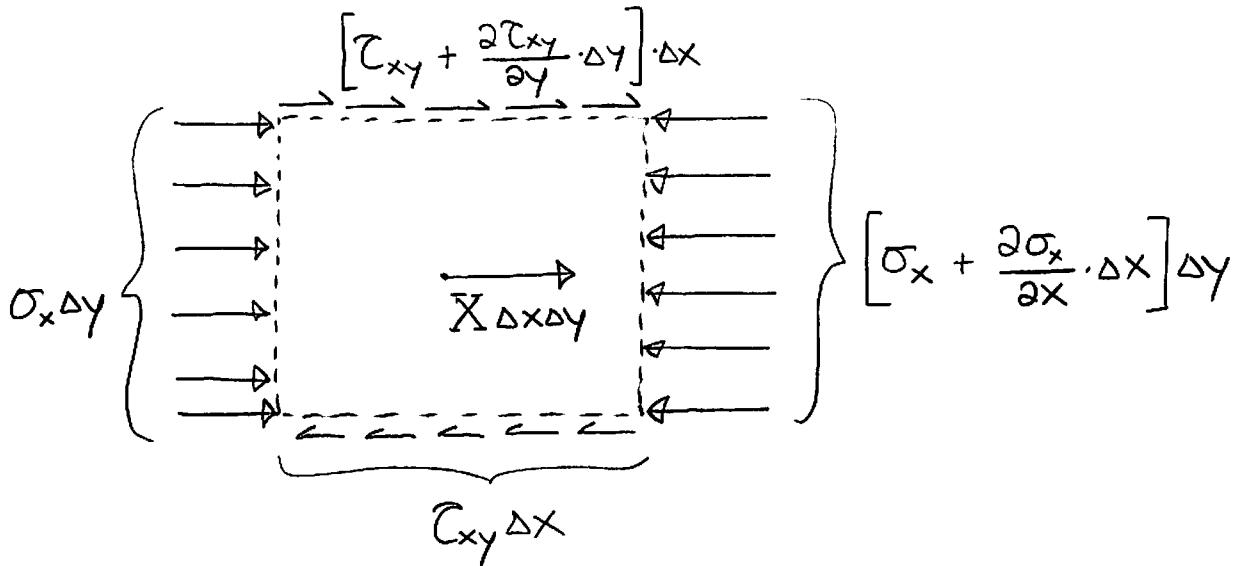
Looking first at the reaction forces due to momentum flow:



$$\begin{aligned} & [\rho uv + \frac{\partial(\rho uv)}{\partial y} \cdot \Delta y] \cdot \Delta x \\ & \rho u^2 \Delta y \\ & \frac{\partial}{\partial t} (\rho u) \cdot \Delta x \Delta y \\ & [\rho u^2 + \frac{\partial(\rho u^2)}{\partial x} \cdot \Delta x] \Delta y \\ & \rho uv \Delta x \end{aligned}$$

*Note, the transient term $(\frac{\partial}{\partial t})$ is negative because we've moved it to the other side of the eqn.

We can do the same for the forces acting on the CV, where we have normal stress (σ_x), tangential stress (τ_{xy}), and x -direction body force per unit volume (X).



Writing out our complete momentum balance:

$$\begin{aligned}
 & -\frac{\partial}{\partial t} (\rho u \Delta x \Delta y) + \rho u^2 \Delta y - \left[\rho u^2 + \frac{\partial}{\partial x} (\rho u^2) \Delta x \right] \Delta y \\
 & + \rho u v \Delta x - \left[\rho u v + \frac{\partial}{\partial y} (\rho u v) \Delta y \right] \Delta x \\
 & + \sigma_x \Delta y - \left[\sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot \Delta x \right] \Delta y - C_{xy} \Delta x \\
 & + \left[C_{xy} + \frac{\partial C_{xy}}{\partial y} \Delta y \right] \Delta x + X \Delta x \Delta y = 0
 \end{aligned}$$

Dividing through by $\Delta x \cdot \Delta y$, & letting $\Delta x, \Delta y \rightarrow 0$

$$\rho \frac{du}{dt} + u \left[\frac{\partial p}{\partial t} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] = - \frac{\partial \sigma_x}{\partial x} + \frac{\partial C_{xy}}{\partial y} + X$$

We know from mass conservation that the term in the brackets is zero.

$$\rho \frac{du}{dt} = - \frac{\partial \sigma_x}{\partial x} + \frac{\partial C_{xy}}{\partial y} + X$$

Now we relate the stresses σ_x and C_{xy} to the local flow field by constitutive relations

$$\sigma_x = \rho - 2\mu \frac{\partial u}{\partial x} + \frac{2}{3} \mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$\sigma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Viscosity, found by experiment.

Scalar viscosity coefficient
found by experiments.

These relations are empirical in origin. We define μ as the fluid viscosity. Note, these are valid for a Newtonian fluid which specifies the following conditions:

- 1) When at rest the stress is hydrostatic and the pressure inside the fluid is the thermodynamic press.
- 2) The stress tensor σ_{ij} is linearly proportional to δ_{ij} (the deformation tensor).
- 3) No shear force may act during solid body rotation.
- 4) There are no preferred directions in the fluid, so fluid properties are point properties.

These approximate water, air, and many other fluids.

Substituting back our constitutive equations:

$$\rho \frac{du}{dt} = - \frac{\partial p}{\partial x} + \frac{2}{\partial x} \left[2\mu \frac{\partial u}{\partial x} - \frac{2\mu}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{2}{\partial y} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + X$$

↳ Navier-Stokes Equation (the real one!)

A useful approximation for us is to assume incompressible and μ is constant. Then:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{from mass conservation of an incompressible fluid})$$

Now the x -momentum equation becomes:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{2}{\partial x} \left[2u \frac{\partial u}{\partial x} - \frac{2u}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{2}{\partial y} \left[u \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]$$

$$= -\frac{\partial p}{\partial x} + 2u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} + u \frac{2}{\partial y} \cdot \frac{\partial v}{\partial x} + X$$

But note: $u \frac{2}{\partial y} \cdot \frac{\partial v}{\partial x} = u \frac{2}{\partial x} \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ (cons. of mass)

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + 2u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} - u \frac{\partial^2 u}{\partial x^2} + X$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + X$$

Expanding the left hand side:

$$\boxed{\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + X}$$

\hookrightarrow x -momentum equation (20).

In general:

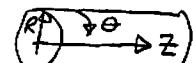
$$\boxed{\rho \frac{\partial v}{\partial t} = -\nabla p + u \nabla^2 v + F}$$

Where F is the body force per unit volume.
Similar equations can be derived for cylindrical & spherical coordinates. For example (you may use this one):

$$\boxed{\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z}}$$

$$\boxed{+ u \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial^2 v_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + Z}$$

\hookrightarrow z -momentum eqn. in cylindrical coord.



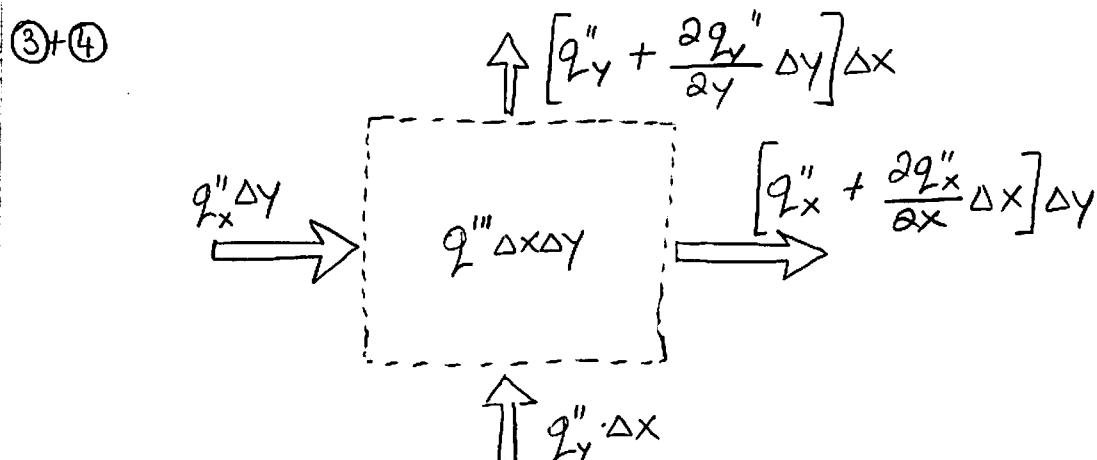
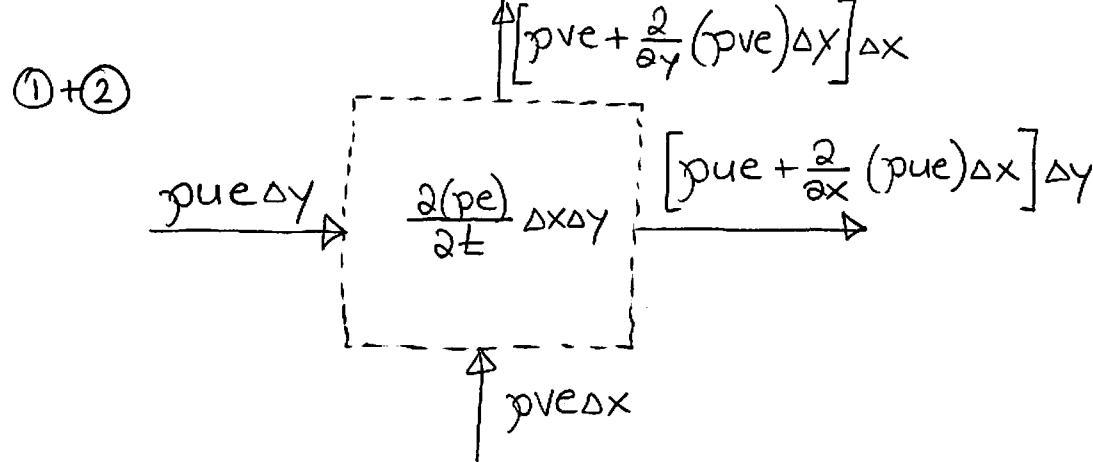
First Law of Thermodynamics

So far, we've equipped ourselves well to solve fluid mechanics problems. However, the heat transfer part of the convection problem requires a solution to the temperature distribution in the flow (especially close to the wall).

To solve, we need the energy equation (1st law)

$$\left(\begin{array}{l} \text{Rate of energy} \\ \text{accumulation in} \\ \text{the CV} \end{array} \right)_1 = \left(\begin{array}{l} \text{net transfer} \\ \text{of energy} \\ \text{by fluid flow} \end{array} \right)_2 + \left(\begin{array}{l} \text{net heat} \\ \text{transfer by} \\ \text{conduction} \end{array} \right)_3 + \left(\begin{array}{l} \text{heat} \\ \text{gener.} \end{array} \right)_4 - \left(\begin{array}{l} \text{work} \\ \text{of CV} \\ \text{on its env.} \end{array} \right)_5$$

Let's draw our CV



Note, I haven't drawn the work contribution to either CV. (11)

So if we write out our terms (1) to (5)

$$(\dots)_v = \Delta x \times \Delta y \frac{2}{\partial E} (\rho e) \quad ; \quad e = \text{specific internal energy}$$

$$(\dots)_2 = - (\Delta x \Delta y) \left[\frac{\partial}{\partial x} (p_{ue}) + \frac{\partial}{\partial y} (p_{ve}) \right]$$

$$(\dots)_3 = -(\Delta x \Delta y) \left(\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} \right)$$

$$(\dots)_4 = (\Delta x \Delta y) q'''$$

$$(\dots)_5 = (\Delta \times \Delta y) \left(O_x \frac{\partial u}{\partial x} - C_{xy} \frac{\partial u}{\partial y} + O_y \frac{\partial v}{\partial y} - C_{yx} \frac{\partial v}{\partial x} \right)$$

$$+ \left\{ (\Delta x \Delta y) \left(u \frac{\partial \alpha_x}{\partial x} - u \frac{\partial \alpha_y}{\partial y} + v \frac{\partial \alpha_y}{\partial y} - v \frac{\partial \alpha_x}{\partial x} \right) \right\}$$

For the work term (s) we had to use our CV from the x-momentum equation.

For example, $W_{\text{LEFT}} = \int \underline{\underline{F}}(\underline{x}, t) \cdot \underline{\underline{\delta y}}(t) dt$

work on the environment
 \downarrow
 $(\underline{\underline{F}} \cdot \underline{\underline{\delta y}})$
 force on
the bound.
 Boundary displacement
per unit time

Similarly, on the right side:

$$W_{RIGHT} = \left(\sigma_x + \frac{\partial \sigma}{\partial x} \Delta x \right) \Delta y \cdot \left(u + \frac{\partial u}{\partial x} \Delta x \right)$$

The net work transfer rate due to these two contributions is

$$\begin{aligned}
 W_{NET} &= \left(\sigma_x \Delta y + \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y \right) \left(u + \frac{\partial u}{\partial x} \Delta x \right) - u \sigma_x \Delta y \\
 &= u \cancel{\sigma_x \Delta y} + \sigma_x \frac{\partial u}{\partial x} \Delta x \Delta y + u \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y + \frac{\partial \sigma_x}{\partial x} \frac{\partial u}{\partial x} \Delta x^2 \Delta y - \cancel{u \sigma_x \Delta y} \\
 &= \sigma_x \frac{\partial u}{\partial x} \Delta x \Delta y + u \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y + \frac{\partial \sigma_x}{\partial x} \cdot \frac{\partial u}{\partial x} \Delta x^2 \Delta y
 \end{aligned}$$

The last term goes to zero when we divide by $\Delta x \Delta y$ and let $\Delta x \rightarrow 0$ & $\Delta y \rightarrow 0$

Similarly, we can do the same procedure for the other stresses.

We can also show that the 4 terms in the $\{\}$ of equat. (5) reduce to: (try deriving it \Rightarrow substitute in $\sigma_x, \sigma_y, C_{xy}, C_{yx}$)

$$\Delta x \Delta y \left(u \frac{\partial \sigma_x}{\partial x} + u \frac{\partial C_{xy}}{\partial y} + v \frac{\partial \sigma_y}{\partial y} - v \frac{\partial C_{yx}}{\partial x} \right) = -\rho \frac{D}{Dt} \left(\frac{u^2 + v^2}{2} \right)$$

This equates to the change in kinetic energy of the fluid packet.

Since $-\rho \frac{D}{Dt} \left(\frac{u^2 + v^2}{2} \right) \ll \frac{2(\rho e)}{Dt}$, we neglect these terms.

Assembling everything together: (back substituting constitutive rel.)

$$\boxed{\rho \frac{De}{Dt} + e \left(\frac{D\rho}{Dt} + \rho \nabla \cdot V \right) = -\nabla \cdot q'' + q'' - \rho \nabla \cdot V + u \Phi}$$

\hookrightarrow 2D energy equation, where:

$q'' \equiv$ heat flux vector (q''_x, q''_y)

Φ = viscous dissipation function

To solve for Φ we need to back substitute our constitutive relations into (5) and simplify.

In cartesian coordinates (30)

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \right. \\ \left. + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2$$

To simplify our lives, we can assume 2D & incompressible

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2$$

Now we can deal with simplifying our other terms.
From Thermodynamics:

$$h = e + \left(\frac{1}{\rho}\right) P \Rightarrow \text{enthalpy}$$

$$\frac{\partial h}{\partial t} = \frac{\partial e}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial t} - \frac{P}{\rho^2} \frac{\partial P}{\partial t} \quad (\text{quotient rule})$$

Also, via Fourier's law of heat conduction

$$q'' = -k \nabla T$$

Back substituting:

$$\rho \frac{\partial h}{\partial t} = \nabla \cdot (k \nabla T) + q''' + \frac{\partial P}{\partial t} + \mu \Phi - \underbrace{\frac{P}{\rho} \left(\frac{\partial P}{\partial t} + \rho \nabla \cdot \nabla \right)}_0$$

Finally, from mass conservation:

$$\rho \frac{\partial h}{\partial t} = \nabla \cdot (k \nabla T) + q''' + \frac{\partial P}{\partial t} + \mu \Phi$$

Note, it's tempting to say $h = c_p T$, however this is only correct for "ideal" gases. More rigorously

$$dh = Tds + \frac{1}{\rho} dP \quad (1) \Rightarrow T = \text{absolute temp.}$$

$ds = \text{specific entropy change}$

We can now write the following:

$$ds = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP \quad (2)$$

From Maxwell's relations

$$\left(\frac{\partial S}{\partial P}\right)_T = - \left[\frac{\partial (1/\rho)}{\partial T} \right]_P = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{\beta}{\rho} \quad (3)$$

where $\boxed{\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_P = \text{coefficient of thermal expansion}} \quad (4)$

Using thermodynamic relations, we can also show that:

$$\left(\frac{\partial S}{\partial T}\right)_P = \frac{C_p}{T} \quad (5)$$

Putting equations (1) to (5) together, we obtain

$$dh = C_p dT + \frac{1}{\rho} (1 - \beta T) dP$$

Back substituting into our energy equation, we obtain

$$\boxed{\rho C_p \frac{\partial T}{\partial E} = \nabla \cdot (k \nabla T) + q'' + \beta T \frac{\partial P}{\partial E} + u \bar{\Phi}}$$

\hookrightarrow Temperature based 1st Law of Thermodynamics
Note, we can simplify this for certain cases

For an ideal gas: $\beta = \frac{1}{T}$

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''' + \frac{\partial P}{\partial t} + u \bar{\Phi}$$

For an incompressible liquid (water): $\beta = 0$ (in real life, $\beta_{\text{water}} \approx 0$)

$$\rho C \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''' + u \bar{\Phi}$$

Note, these are general solutions but for most of the problems we'll solve in this course, we will have:

- 1) $k = \text{constant}$ (constant properties)
- 2) $q''' = 0$ (no internal energy generation)
- 3) $u \bar{\Phi} \approx 0$ (negligible viscous dissipation)
- 4) $\beta T \frac{\partial P}{\partial T} \approx 0$ (negligible compressibility effects)

So our formulation boils down to:

$$\rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T$$

For cartesian (x, y, z)

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

For cylindrical (r, θ, z)

$$\rho C_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right.$$

For spherical (r, ϕ, θ):

$$\left. + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

$$\rho C_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \dots \right]$$

Note, for extremely viscous fluids & flows (i.e. lubrication problems or piping of crude oil), the viscous dissipation must be taken into account:

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + u \Phi$$

For exact definitions of Φ in other coordinate systems, see pg. 14 of Bejan (4th edition).

When the energy equation was first developed by Fourier & Poisson, they always used c instead of c_p . This is ok since they were working with incompressible fluids ($c=c_p$). However, c_v should never be used as it ignores the PdV work done by the moving fluid packet as it expands and contracts.

Second Law of Thermodynamics

We don't need the second law of thermo to solve convection problems. Where it becomes important is when we design systems that have convective heat transfer with a fluid.

The second law states all processes are irreversible:

$$\frac{dS_{cv}}{dt} \geq \sum_i \frac{q_i}{T_i} + \sum_{inlet} \dot{m}s - \sum_{outlet} \dot{m}s$$

where; S_{cv} = instantaneous entropy in the control volume
 q_i & T_i = heat transfer and absolute temperature of the boundary i . (+ means inflow)

The measure of the irreversibility of the process is quantified by the strength of the inequality, or the entropy generation rate S_{gen} :

$$S_{gen} = \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{inlet} \dot{m}_s + \sum_{outlet} \dot{m}_s \geq 0$$

We also know from exergy analysis that

$$W_{lost} = T_0 S_{gen} \quad \text{= destruction of useful work due to entropy generation.}$$

T_0 = absolute temp. of ambient reservoir.

This shows the engineering importance of making the convection process as efficient as possible and estimating S_{gen} for a fluid flow.

Based on analysis similar to previously done, we can say that the entropy generated at an arbitrary point in a flow per unit time per unit volume, S''_{gen} is:

$$S''_{gen} = \underbrace{\frac{k}{T^2} (\nabla T)^2}_{\geq 0} + \underbrace{\frac{\mu}{T} \overline{\Phi}}_{\geq 0} \geq 0$$

\Rightarrow 1st term represents entropy gen. due to temp. gradients, while second term represents viscous dissipation.

In a 2D convective field, the local S''_{gen} is: (pressure drop)

$$S''_{gen} = \frac{k}{T^2} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{\mu}{T} \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \geq 0$$

Note, this is very useful for entropy generation minimization which is a whole field in itself. We can design our flow geometry such that S_{gen} is minimized along with W_{lost} . (18)

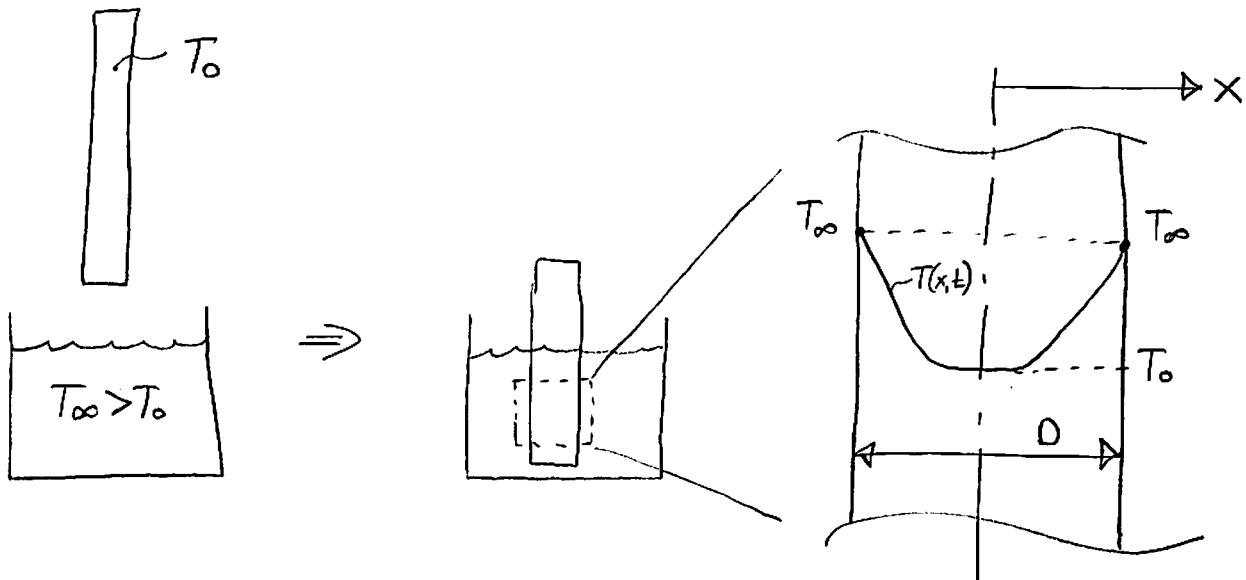
Scaling (or Scale Analysis)

This method will provide most information per unit of intellectual effort.

Different from dimensional analysis (Buckingham Π theorem) as you usually know the dimensional differential equations governing the problem at hand.

Where the two meet is when determining key parameters.

Let's do an example to see how scaling works. Consider a metal bar plunged into a hot fluid.



Assuming that the bar surface temperature assumes the fluid temperature instantaneously. What is the time required for the thermal front to penetrate the plate, i.e. the time for the center of the plate to "feel" the heating imposed on the outer surfaces.

Note, for $T(x=\frac{D}{2}, t=0) = T_\infty$ to be true, is B_i low or high?

$$\text{Here } Bi = \frac{\text{conduction resistance}}{\text{convection resistance}} = \frac{L/kA}{1/hA} = \frac{hL}{k_{\text{bar}}}$$

We know if $Bi < 0.1$, $T_{\text{bar}} \neq f(x)$ & only $f(t)$, hence $Bi > 0.1$ or h is very high.

Remember, $q = \frac{\Delta T}{R}$, so for $q = \text{constant}$ $\frac{\Delta T_{\text{conv}}}{\frac{1}{hA}} = \frac{\Delta T_{\text{cond}}}{\frac{L}{kA}}$

$$h\Delta T_{\text{conv}} = k \frac{\Delta T_{\text{cond}}}{L}$$

So for h very high, $\Delta T_{\text{conv}} \ll \Delta T_{\text{cond}}$ & $\boxed{T(x=\frac{D}{2}, t) = T_{\infty}}$
QED

Now back to our problem. We can begin by writing our energy equation in cartesian:

$$\rho C_p \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = h \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q''' + \beta T \frac{\partial P}{\partial t} + u \Phi$$

We can drop many of our terms since its a stationary CV
 $u = v = w = 0$,

Also, no temperature gradients in the y, z direction. Hence:

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (1)$$

Scaling requires us to estimate the order of magnitude of each term appearing in our differential equation (1)

LHS: $\rho C_p \frac{\partial T}{\partial t} \Rightarrow \rho \sim \rho$ (Notation: " \sim " = is of the same order of magnitude as)
 $C_p \sim C_p$
 $\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t} \sim \frac{T_{\infty} - T_0}{t}$

$$\rho C_p \frac{\partial T}{\partial t} \sim \rho C_p \frac{\Delta T}{t}$$

RHS: $k \sim k$

$$\frac{\partial^2 T}{\partial x^2} = \frac{2}{2x} \left(\frac{\partial T}{\partial x} \right) \sim \frac{1}{0/2} \cdot \frac{\Delta T}{0/2}$$

$$k \frac{\partial^2 T}{\partial x^2} \sim k \frac{\Delta T}{(0/2)^2}$$

Now equating our two terms:

$$\rho C_p \frac{\Delta T}{t} \sim k \frac{\Delta T}{(0/2)^2} \Rightarrow t \sim \frac{(0/2)^2 \rho C_p}{k} \sim \frac{(0/2)^2}{\alpha}$$

$$\alpha = \frac{k}{\rho C_p} \equiv \text{thermal diffusivity}$$

So very simple scaling has shown us that:

$$D \sim \sqrt{\alpha t} \Rightarrow \text{where } D = \text{thermal penetration depth.}$$

Rules of Scaling

- ① Define the spatial extent of the region in which scaling is performed (this is the common thread to dimensional analysis). For example, we just used $0/2$ for the bar problem. For a boundary layer problem, lengths like L or δ are important.
- ② Any equation constitutes an equivalence between the scales of the two dominant terms appearing in the equation. From our last example, the eqn. has only 2 terms, hence they must be same order of magnitude. In general, eqn's can have more terms, not all of them important.

③ If we sum two terms (a & b)

$$c = a + b$$

and the order of one term is greater than the other

$O(a) > O(b) \Rightarrow O(\dots)$ denotes "order of magnitude"

then the order of the sum is dominated by the dominant term:

$$O(c) \sim O(a)$$

④ If we sum two terms (a & b)

$$c = a + b$$

and the two terms are the same order of magnitude

$$O(a) \sim O(b)$$

then the sum is also of the same order of magnitude

$$O(c) \sim O(a) \sim O(b)$$

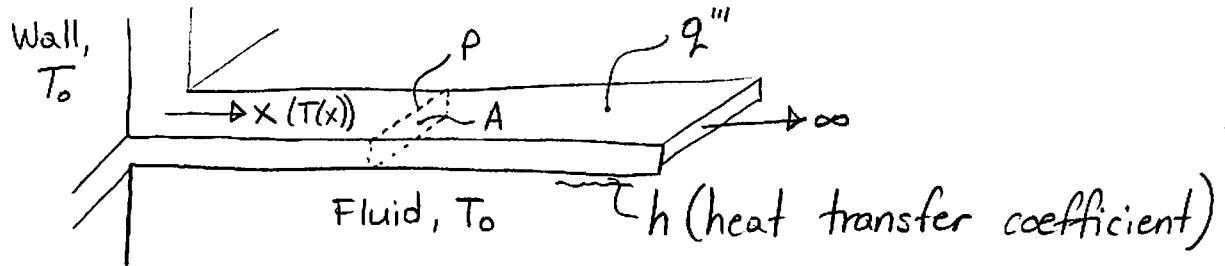
⑤ In any product: $p = ab$

The order of magnitude of the product is equal to the product of the orders of magnitude:

$$O(p) \sim O(a) \cdot O(b)$$

If ratio: $r = \frac{a}{b}$, then $O(r) \sim \frac{O(a)}{O(b)}$

Example | Scaling with a fin with heat generation.



Writing out our energy equation, we will obtain:

$$kA \frac{\partial^2 T}{\partial x^2} - hP(T - T_0) + q'''A = 0$$

- a) Let T_∞ be the fin temperature sufficiently far from the wall. Show if x is large enough that longitudinal conduction becomes negligible.

$$kA \frac{\partial^2 T}{\partial x^2} \sim kA \frac{\Delta T}{\delta^2} \quad \text{where } \Delta T = T - T_0$$

$$hP(T - T_0) \sim hP\Delta T$$

$$q'''A \sim q'''A$$

Our equation becomes:

$$kA \frac{\Delta T}{\delta^2} - hP\Delta T + q'''A = 0$$

Normalizing by the conduction term $(kA \frac{\Delta T}{\delta^2})$

$$\frac{kA \frac{\Delta T}{\delta^2}}{kA \frac{\Delta T}{\delta^2}} - \frac{hP\Delta T}{kA \frac{\Delta T}{\delta^2}} + \frac{q'''A}{kA \frac{\Delta T}{\delta^2}} = 0$$

$$1 - \frac{hP\delta^2}{kA} + \frac{q'''A}{k\Delta T} = 0 \Rightarrow \text{Hence as } \delta \rightarrow \infty, \text{ the first term becomes negligible.}$$

b) Using scaling determine the fin temperature far from the wall.

We just showed conduction is negligible there, hence:

$$hP\Delta T \sim q'''A$$

$$T_\infty - T_0 \sim \frac{q'''A}{hP}$$

c) How far away from the wall is the heat transfer dominated by a balance of heat conduction & heat generation only?

We have from our equation in (a) that:

$$1 - \frac{hPS^2}{kA} + \frac{q'''S^2}{k\Delta T} = 0$$

$\underbrace{\frac{hPS^2}{kA}}$ negligible compared to 1 or $\frac{hPS^2}{kA} \ll 1$

$$S \ll \sqrt{\frac{kA}{hP}}$$

Another way to do this is to say near the wall:

$$1 \sim \frac{q'''S^2}{k\Delta T}$$

$$S \sim \sqrt{\frac{k\Delta T}{q'''}}$$

\Rightarrow Not the best way since $\Delta T = T - T_0$

Note if we balance the two:

$$\sqrt{\frac{k\Delta T}{q'''}} \ll \sqrt{\frac{hA}{hP}} \Rightarrow \frac{\Delta T}{q'''} \ll \frac{A}{hP} \text{ or } h \ll \frac{q'''A}{P\Delta T}$$

Makes sense, if P is large, inequality is hard to obtain.

Dimensional Analysis (Buckingham Π Theorem)

Allows us to find the interdependence of a system's physical variables by consideration of appropriate balances, (i.e. forces, energies, etc...)

\Rightarrow Requires physical insight, a knowledge of the physics and experience in the field.

The theorem: For a system with M physical variables (e.g. density, speed, length, viscosity) describable in terms of N fundamental units (i.e. mass, length, time, temperature), there are $M-N$ dimensionless groups.

① If $M-N = P$, then the system is characterized by expressing the interdependency of the P dimensionless groups.

② If $M-N = 1$, then the single dimensionless parameter must be constant Π

Obtaining Dimensionless Groups

① List all physical variables (M) and fundamental units (N) and identify their dimensional form.

Denote "dimensions of x " by $[x]$

e.g.
$$\frac{[v]}{\text{velocity}} = \frac{L}{T}, \frac{[g]}{\text{gravity}} = \frac{L}{T^2}, \frac{[\rho]}{\text{density}} = \frac{M}{L^3}, \frac{[F]}{\text{force}} = \frac{ML}{T^2}$$

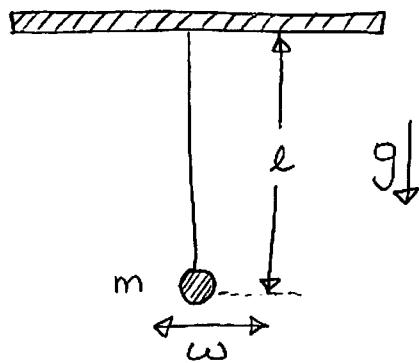
② Determine the number of dimensionless groups ($M-N$)

③ Form a set of $(M-N)$ dimensionless groups by assuming arbitrary exponents for each of the variables.

Be sure that each of the physical variables appears in at least one group.

This last part is where intuition is very useful.

Example #1] Pendulum (with small amplitude θ)



Physical Variables: m, l, g, ω
Fundamental Units: M, L, T

Buckingham Theorem: $M - N = 4 - 3 = 1$ independent dimensionless group

Now we can solve:

$$\pi = \omega \cdot l^a \cdot g^b \cdot m^c$$

$$[\omega] = \frac{1}{T}, \quad [l] = L, \quad [g] = \frac{L}{T^2}, \quad [m] = M$$

$$[\pi] = \frac{1}{T} \cdot L^a \cdot \frac{L^b}{T^{2b}} \cdot M^c = 1 \quad (\text{since only 1 dimensionless group})$$

$$T: 2b + 1 = 0 \Rightarrow b = -\frac{1}{2}$$

$$L: a + b = 0 \Rightarrow a = \frac{1}{2}$$

$$M: c = 0$$

$$\boxed{\pi = 1 = \omega \left(\frac{l}{g} \right)^{1/2}}$$

Now we can say:

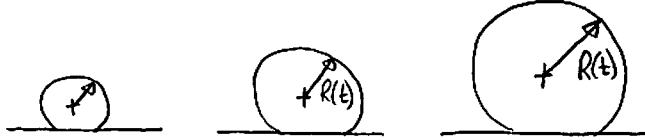
$$\boxed{\omega \sim \left(\frac{g}{l} \right)^{1/2}}$$

Scaling (this is where the two meet)

Example #2 / Atom-bomb blast cloud

What is the time-dependence of the radius of the blast cloud following the detonation of an atom bomb?

i.e. What is $R(t)$?



Physical variables: energy released, E
radius, R
time, t
air density, ρ

Aside:

Why not viscosity, μ ? Consider the length & time scales.
Inertia dominates any viscous effects, so no need to consider it. We could have included it, but it would end up coming out as $Re_R \gg 1$

Fundamental units: $M, L, T \Rightarrow N = 3 \Rightarrow 1$ dimensionless group

$$[E] = \frac{ML^2}{T^2}, [R] = L, [\rho] = \frac{M}{L^3}, [t] = T$$

$$\Pi = E \cdot t^\alpha \cdot \rho^b \cdot R^c = \frac{ML^2}{T^2} \cdot T^\alpha \cdot \frac{M^b}{L^3} \cdot L^c$$

$$\left. \begin{array}{l} M: 1+b=0 \Rightarrow b=-1 \\ L: 2+c=3b \Rightarrow c=-5 \\ T: \alpha-2=0 \Rightarrow \alpha=2 \end{array} \right\} \quad \boxed{\Pi = \frac{Et^2}{\rho R^5} = \text{constant}}$$

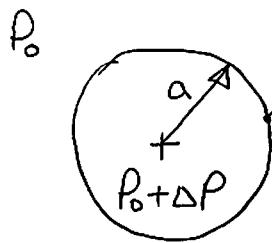
$$\therefore R \sim \left(\frac{E}{\rho} \right)^{1/5} \cdot t^{2/5}$$

$$R(t) = C \left(\frac{E}{\rho} \right)^{1/5} t^{2/5}$$

constant of proportionality

G.I. Taylor did this scaling for an atom bomb blast in 1945 & published it, finding the energy of the blast. He was arrested!

Example #3 | What is the pressure inside a bubble?



$$\sigma = \text{surface tension} \quad [\sigma] = \frac{\text{Force}}{\text{Length}} = \frac{M}{T^2}$$

Physical variables: $\Delta P, \sigma, a \Rightarrow$ note ΔP counts as 2 variables
Fundamental units: M, L, T

$$[\Delta P] = \left[\frac{F}{L^2} \right] = \frac{ML}{T^2} \cdot \frac{1}{L^2}$$

$$[\sigma] = \left[\frac{F}{L} \right] = \frac{M}{T^2}$$

$$[a] = L$$

$$\pi = \frac{\Delta P \cdot a}{\sigma} = \text{constant}$$

$$\boxed{\Delta P \sim \frac{\sigma}{a}}$$

$\rho = M - N = 1 \Rightarrow$ Means 1 dimensionless group will exist.

So small bubbles will have higher pressure. This is why champagne is louder than beer. This is also how you tell the quality of a good champagne. Bubbles will rise slowly & pop loudly, since good champagnes are high pressure.

So far, these examples are meant to show you how important it is to choose your physical variables correctly. This also applies to scaling analysis.

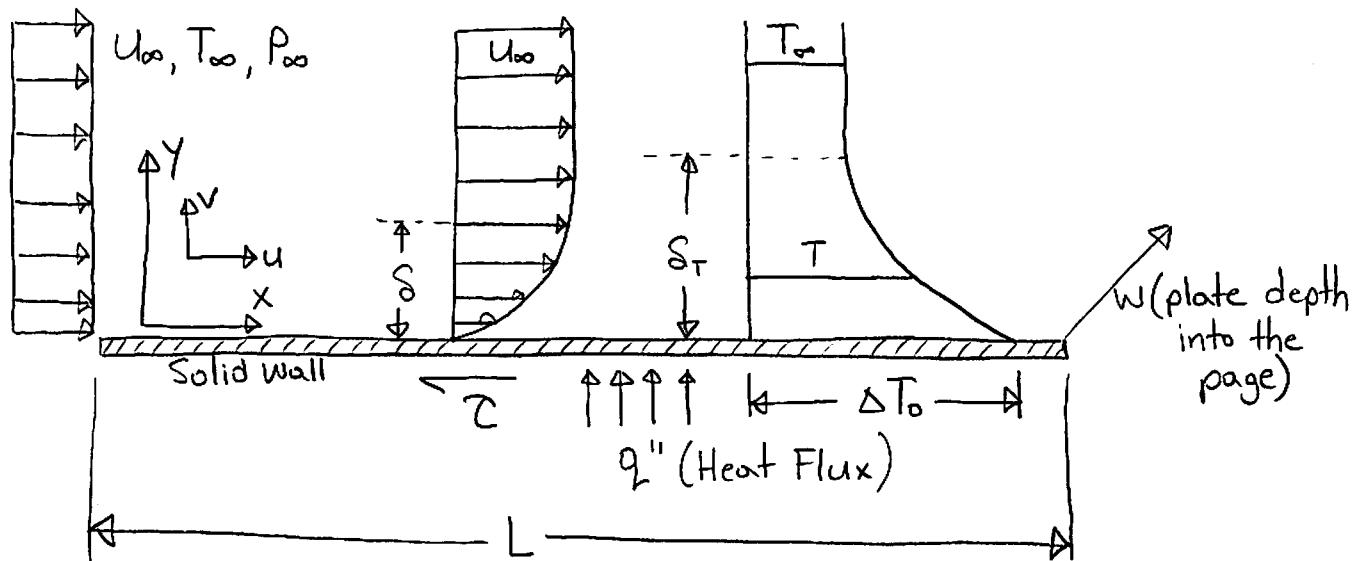
For a more formal proof of why Buckingham π theorem works, look at rank-nullity theorem.

Laminar Boundary Layer Flow

So far, we have been developing tools to analyze convection problems.

Convection = advection + diffusion

Let's consider the simplest possible problem to analyze:



As engineers, we want to know:

- 1) The net force exerted by the stream on the plate
- 2) The resistance to heat transfer from the plate to stream.

We know we need the following: ($C = \text{skin friction}$)

$$F = \int_0^L C w dx \quad \text{and} \quad q = \int_0^L q'' w dx$$

For the Newtonian fluids we have been dealing with so far:

$$C = \mu \left(\frac{\partial u}{\partial y} \right) \Big|_{y=0}$$

$$\text{and } q'' = h(T - T_0)$$

Heat Transfer coeff.

Aside:

Note, many people attribute $q = h\Delta T$ to Newton as his "law of cooling". However, this is not true. Newton found the following:

$$\frac{dT}{dt} \sim \Delta T \Rightarrow \text{At that time, there was no concept of } h \text{ or } C_p.$$

The real person to write it as $q = h\Delta T$ was Fourier!

At this point, we can accept empirically that the no slip condition at the wall is true (i.e. $U(y=0) = 0$). Therefore, in the fluid layer immediately adjacent to the wall, we have pure conduction:

$$q'' = -k_f \left(\frac{\partial T}{\partial y} \right) \Big|_{y=0} \quad (\text{note, heat flux is positive when the wall heats the stream})$$

Hence we can now write:

$$h = \frac{-k_f \left(\frac{\partial T}{\partial y} \right) \Big|_{y=0}}{T_0 - T_\infty}$$

So we see that in order to solve for F & q , we must first solve for the velocity and temperature fields adjacent to our wall: (U, V, T) .

But before we do that, it's important to consider:

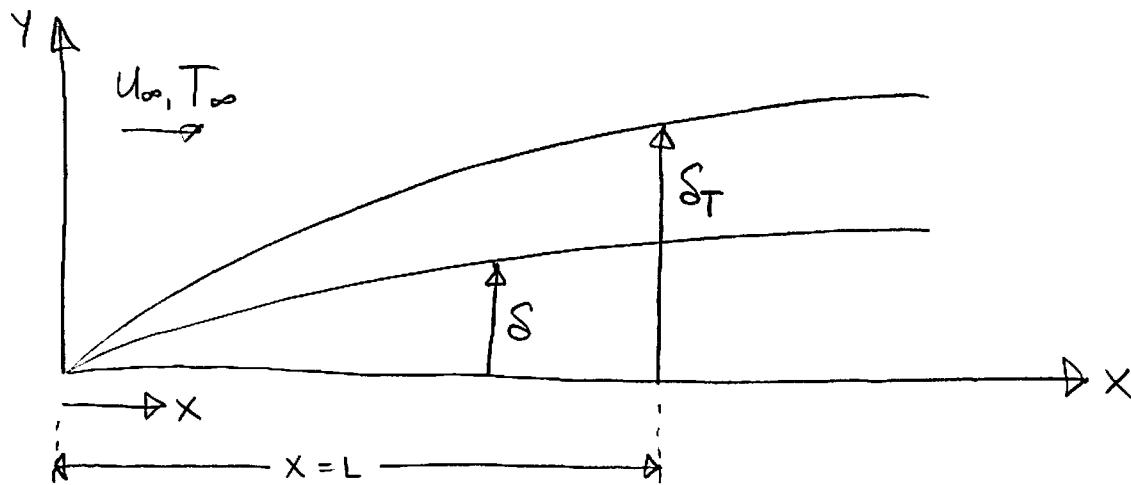
$$\frac{h}{k_f} = \frac{-\left(\frac{\partial T}{\partial y}\right) \Big|_{y=0}}{T_0 - T_\infty} \Rightarrow \text{Making this dimensionless} \cdot \frac{L}{L}$$

$$\frac{hL}{k_f} = \frac{-\left(\frac{\partial T}{\partial y}\right) \Big|_{y=0}}{\left(\frac{T_0 - T_\infty}{L}\right)} = Nu_L = \text{Nusselt Number}$$

In this case, we can see that:

$$Nu_L = \frac{\text{Heat transfer due to convection (advection+diffusion)}}{\text{Heat transfer due to conduction only (diffusion)}}$$

Note, the dimension L is chosen to be the distance in the direction of flow and is somewhat arbitrary since the denominator is not a "real" heat transfer due to conduction only.



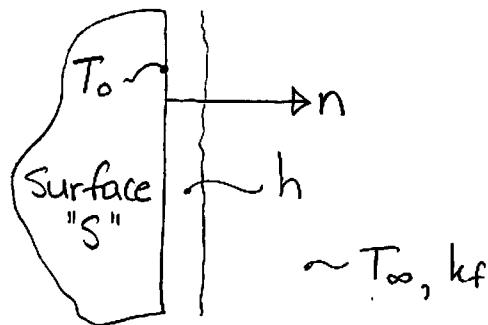
So when you say: $Nu_L = \frac{hL}{k_f} = \frac{\text{convection h.t. at location } L}{\text{conduction h.t. of a water slab of thickness } L}$

Many books say Nu is ratio of convection to conduction if the flow would stop, but this is incorrect since it would then become a transient problem. This is why sometimes Biot number is much easier to understand because it usually deals with finite shapes:

$$Bi = \frac{h\alpha}{k_{\text{body}}} = \frac{\text{conduction resistance inside the body}}{\text{convection resistance outside the body}}$$

α = body dimension.

So let's see if we can get any more meaning out of Nu.



$$q'' = h(T_0 - T_\infty) \quad \textcircled{1}$$

or

$$q'' = -k_f \frac{\partial T}{\partial n} \Big|_S \quad \textcircled{2}$$

Let's non-dimensionalize:

$$\Theta = \frac{T - T_\infty}{T_0 - T_\infty}, \quad n^* = \frac{n}{L} = \text{where } L \text{ is some arbitrary length along the plate or surface "S".}$$

Back substituting into \textcircled{1} & \textcircled{2} and equating them:

$$-k_f \frac{(T_0 - T_\infty)}{L} \cdot \frac{\partial \Theta}{\partial n^*} \Big|_S = h(T_0 - T_\infty)$$

$$\boxed{-\frac{\partial \Theta}{\partial n^*} \Big|_S = \frac{hL}{k_f} = Nu_L = \text{Nusselt Number}}$$

So we can see that the Nusselt number also physically represents a non-dimensional temp. gradient at the surface. The larger the gradient, the larger the heat transfer.

In general, as we already showed, in order to solve for h , we first need u, v, T . Note (u, v) and (T) are coupled and we need to solve both the momentum & energy equations.

Also: $Nu \sim h = f(u, v, T)$, hence: $Nu = f(\text{Flow conditions, geometry, fluid properties, \& b.c.'s})$

So now we have all the tools to start solving for C & h . Modeling our flow as incompressible & constant property:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{Mass conservation})$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (\text{x-momentum})$$

$$U \frac{\partial V}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{1}{P} \frac{\partial P}{\partial Y} + U \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) \quad (\gamma\text{-momentum})$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (\text{energy})$$

We have 4 unknowns (u, v, P, T) and the following b.c.'s

i) No slip $\Rightarrow u = 0$
ii) Impermeability $\Rightarrow v = 0$
iii) Wall temp $\Rightarrow T = T_0$

iv) Uniform flow $\Rightarrow U = U_\infty$
 v) Uniform flow $\Rightarrow V = 0$
 vi) Uniform temperature $\Rightarrow T = T_\infty$
}

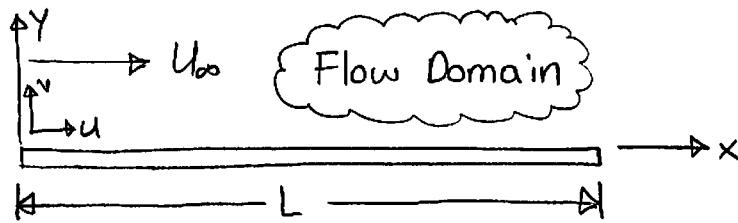
Infinitely far from the wall, in both x & y directions

This formulation is nice, but too cumbersome to solve without a computer, so we'll need to simplify it. To do this, we will use scaling!

The way to simplify our analysis is to realize that there is a thin region close to the plate that contains all the 'action'. Beyond this region, the free stream fluid cannot tell whether the plate is even there.

This concept was developed by Prandtl in early 1900's & it solved a very old problem called D'Alembert's paradox.

Basically, if you non-dimensionalize the momentum equations:



Let's work with x-momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$$\bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}, \quad \bar{p} = \frac{p}{\rho U_\infty^2}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}$$

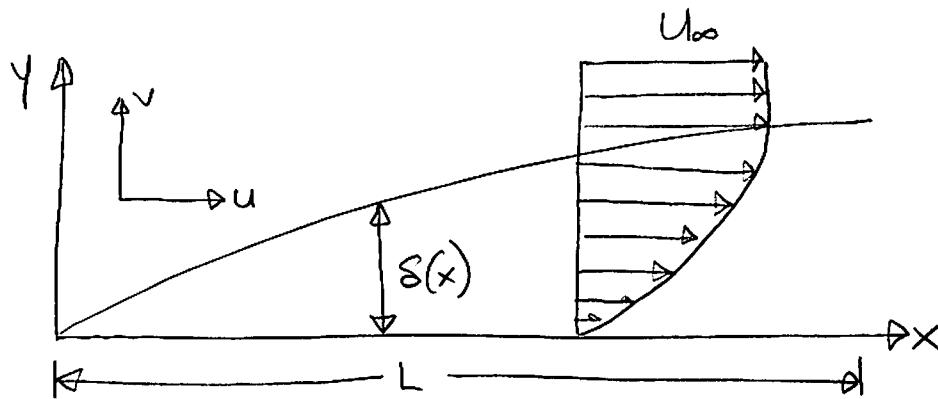
Our dimensionless x-momentum equation becomes:

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = - \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re_L} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$$

where $Re_L = \frac{\rho U_\infty L}{\mu}$ = Reynolds number = $\frac{\text{inertial forces}}{\text{viscous forces}}$.

So we see as $Re_L \rightarrow \infty$, viscous forces (drag) $\rightarrow 0$. This is the source of D'Alembert's paradox.

To fix this problem, Ludwig Prandtl in the early 1900's experimentally realized that there is a very thin region adjacent to the plate where high velocity gradients exist. He called this region the boundary layer.



To solve D'Alembert's paradox, we realize that we've done our scaling incorrectly. To solve the problem we must do our scaling within the boundary layer, where the flow "feels" the plate.

$$u \sim U_\infty, \quad x \sim L, \quad y \sim \delta; \text{ where } \delta \ll L$$

$$\underbrace{u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}}_{\text{Inertia}} = - \underbrace{\frac{p}{\rho} \frac{\partial p}{\partial x}}_{\text{Pressure}} + \underbrace{v \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]}_{\text{Viscosity}}$$

$$\underbrace{U_\infty \frac{u}{L}}_{\text{Inertia}} \quad \underbrace{v \frac{U_\infty}{\delta}}_{\text{Viscosity}} \quad \underbrace{- \frac{p}{\rho L}}_{\text{Pressure}} \quad \underbrace{v \frac{U_\infty}{L^2}}_{\text{Viscosity}} \quad \underbrace{v \frac{U_\infty}{\delta^2}}_{\text{Viscosity}}$$

$$\text{From continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_\infty}{L} + \frac{v}{\delta} = 0 \Rightarrow v \sim U_\infty \left(\frac{\delta}{L} \right)$$

Back substituting into our momentum scaling:

$$\frac{U_\infty^2}{L} = - \frac{p}{\rho L} + v \left[\frac{U_\infty}{L^2} + \frac{U_\infty}{\delta^2} \right]$$

Dividing our viscous terms to find the dominant one:

$$\sqrt{\frac{U_\infty^2}{L^2}} / \sqrt{\frac{U_\infty}{\delta^2}} = \left(\frac{\delta}{L} \right)^2 \ll 1 \Rightarrow \text{Hence only the second viscous term is important.}$$

So from scaling, we reduce our x-momentum eqn. to:

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + V \frac{\partial^2 U}{\partial y^2} \quad (1)$$

Now lets work with our y-momentum equation: ($V \sim \frac{S}{L} U_\infty$)

$$U \underbrace{\frac{\partial V}{\partial x}}_{U_\infty \frac{S}{L} U_\infty \frac{1}{L}} + V \underbrace{\frac{\partial V}{\partial y}}_{U_\infty^2 \left(\frac{S}{L}\right)^2 \frac{1}{S}} = - \frac{1}{\rho} \underbrace{\frac{\partial P}{\partial y}}_{\frac{P}{\rho S}} + V \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right]$$

$$U_\infty \frac{S}{L} U_\infty \frac{1}{L} + U_\infty^2 \left(\frac{S}{L}\right)^2 \frac{1}{S} \sim \frac{P}{\rho S} + V \frac{S}{L} U_\infty \frac{1}{L^2} + V \frac{S}{L} U_\infty \frac{1}{S^2}$$

Rearranging our terms

$$\frac{U_\infty^2}{L} \left(\frac{S}{L}\right) + \frac{U_\infty^2}{L} \left(\frac{S}{L}\right) \sim \frac{P}{\rho S} + V \frac{U_\infty}{L} \cdot \frac{S}{L^2} + V \frac{U_\infty}{L} \frac{1}{S}$$

Let's check which viscous term dominates

$$\frac{V \frac{U_\infty}{L} \cdot \frac{S}{L^2}}{V \frac{U_\infty}{L} \cdot \frac{1}{S}} = \left(\frac{S}{L}\right)^2 \ll 1 \Rightarrow \text{So our second term dominates}$$

Now we need to look at which term dominates.

Re-scaling our pressure term as:

$$P \sim \rho U_\infty^2 \quad (\text{note we don't use } \rho V^2 \text{ since } V \sim \left(\frac{S}{L}\right) U_\infty)$$

Our scaled equation becomes:

$$\underbrace{\frac{U_\infty^2}{L} \left(\frac{S}{L}\right)}_{\text{Inertia (I)}} \sim \underbrace{\frac{U_\infty^2}{S}}_{\text{Pressure (P)}} + \underbrace{V \frac{U_\infty}{L} \frac{1}{S}}_{\text{Viscosity (V)}}$$

$$\frac{I}{P} \sim \frac{U_\infty^2}{L} \left(\frac{S}{L}\right) \cdot \frac{S}{U_\infty^2} \sim \left(\frac{S}{L}\right)^2 \ll 1 \quad (\text{Inertia is negligible compared to pressure})$$

$$\frac{V}{P} \sim V \frac{U_\infty}{L} \cdot \frac{1}{S} \cdot \frac{S}{U_\infty^2} \sim V \frac{1}{L} \quad (\text{Cannot neglect viscosity})$$

Hence our y-momentum equation can be written as:

$$0 = -\frac{1}{P} \frac{\partial P}{\partial y} + V \frac{\partial^2 V}{\partial y^2} \quad (2)$$

Note, equations ① and ② are still too cumbersome to solve analytically. We can look at our pressure terms to help us. Comparing $\frac{\partial P}{\partial x}$ with $\frac{\partial P}{\partial y}$ using scaling:

$$\frac{\partial P}{\partial x} \sim \frac{\rho U_\infty^2}{L} \quad \text{or} \quad \frac{\partial P}{\partial x} \sim \frac{\mu U_\infty}{S^2} \quad (\text{either one is valid as long as we are consistent in both})$$

$$\frac{\partial P}{\partial y} \sim \frac{\mu V}{S^2} \sim \frac{\mu U_\infty}{S^2} \left(\frac{S}{L} \right)$$

Dividing through to see which dominates:

$$\frac{\frac{\partial P}{\partial y}}{\frac{\partial P}{\partial x}} \sim \frac{\mu U_\infty}{S^2} \cdot \left(\frac{S}{L} \right) \cdot \frac{S^2}{\mu U_\infty} \sim \left(\frac{S}{L} \right) \ll 1$$

So from this, we can say that pressure variations in the x-direction dominate the pressure variations in the y-direction.

Hence: $P = f(x)$ only ③

This implies that inside the boundary layer, the pressure at any x is approximately equal to the pressure outside of it.

$$\frac{\partial P}{\partial x} = \frac{\partial P_\infty}{\partial x} \quad (4)$$

Back substituting ④ into ①

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial P_\infty}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} \quad (5)$$

↳ Boundary Layer momentum equation.

We can stipulate our $\partial P_\infty / \partial x$ term by taking a streamline in the free stream (Bernoulli equation). Here we have inviscid flow, (effects of viscosity are negligible).

$$\frac{\partial}{\partial x} \left(P_\infty + \frac{1}{2} \rho U_\infty^2 \right) = \text{constant}$$

$$\frac{\partial P_\infty}{\partial x} + \frac{1}{2} \rho U_\infty \frac{\partial (U_\infty^2)}{\partial x} = 0$$

$$\frac{\partial P_\infty}{\partial x} + \frac{1}{2} \rho U_\infty \frac{\partial U_\infty}{\partial x} = 0$$

$$\frac{\partial P}{\partial x} = - \rho U_\infty \frac{\partial U_\infty}{\partial x} \quad (6)$$

From continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (7)$$

Equations (5)-(7) constitute the general boundary layer equations in 2-D and in cartesian co-ordinates.

In general, there are 4 methods to solve these equations for simplified cases:

- 1) Similarity Solution
- 2) Momentum Integral Methods
- 3) Scaling
- 4) Computational (we will not do this method in this class)

Laminar External Flow over a Flat Plate

With flow over a flat plate, we know that $U_\infty = \text{constant}$.
Therefore:

$$\frac{\partial P}{\partial x} = -\rho_\infty U_\infty \frac{\partial U_\infty}{\partial x} \stackrel{U_\infty = 0}{=} 0$$

Our boundary layer equations become:

$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = V \frac{\partial^2 U}{\partial y^2}$	⑧ ⇒ Momentum
$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$	⑨ ⇒ Continuity

Our boundary conditions are:

$$U = V = 0 \text{ at } y = 0$$

$$U = U_\infty \text{ at } y \rightarrow \infty$$

We can also at this point write out our boundary layer energy equation:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right]$$

Similar to our previous analysis, we can simplify with scaling:

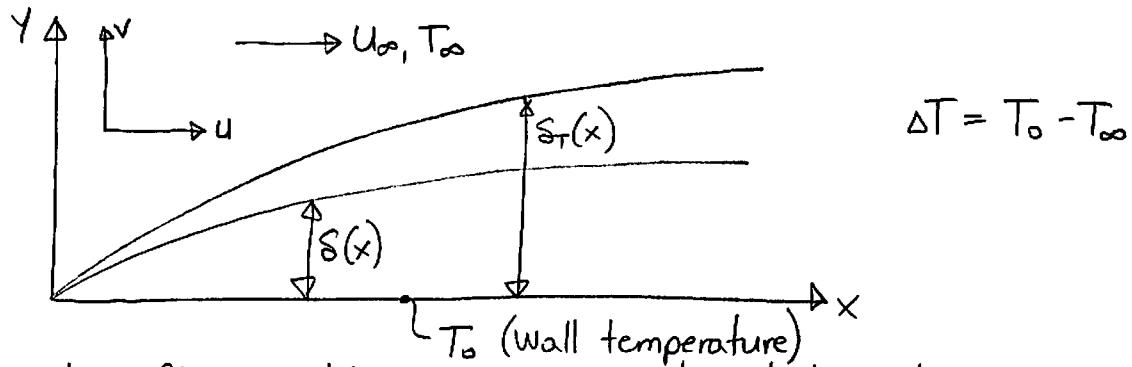
$$\left. \begin{aligned} \frac{\partial T}{\partial x^2} &\sim \frac{\Delta T}{L^2} \\ \frac{\partial^2 T}{\partial y^2} &\sim \frac{\Delta T}{\delta_T^2} \end{aligned} \right\} \left. \begin{aligned} \frac{\partial^2 T}{\partial x^2} &\sim \frac{\Delta T}{L^2} \\ \frac{\partial^2 T}{\partial y^2} &\sim \frac{\Delta T}{\delta_T^2} \end{aligned} \right\} \sim \left(\frac{\delta_T}{L} \right)^2 \ll 1$$

Hence, our boundary layer energy equation becomes:

$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$	⑩
--	---

Note here, $\delta \neq \delta_T$. Both are small compared to L.

Scale Analysis



For the flow problem, we are interested in shear, τ

$$C = \mu \frac{\partial u}{\partial y} \Big|_{y=0} \sim \mu \cdot \frac{U_\infty}{S}$$

To solve for C , we must estimate S using scaling.
From equation ⑧:

$$\underbrace{U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y}}_{\frac{U_0^2}{L}} = \underbrace{U \frac{\partial^2 U}{\partial y^2}}_{\frac{U_0}{S^2}}$$

Inertia Friction (Viscosity)

So from scaling, we can say that in the boundary layer, the inertial forces scale with the friction forces (are of the same order of magnitude)

$$\frac{U_\infty^2}{L} \sim U \frac{U_\infty}{\delta^2} \Rightarrow \delta \sim \left(\frac{UL}{U_\infty} \right)^{1/2} \text{ or } \frac{\delta}{L} \sim Re_L^{-1/2} \quad (11)$$

where $Re_L = U_\infty L / \nu$.

Note, we've defined our boundary layer analysis based on the fact that $\frac{S}{\delta} \ll 1$, hence, we can now check this condition.

So b.l. analysis is valid as long as $Re_L^{1/2} \gg 1$

So now we can solve for our shear stress τ

$$\tau \sim \mu \frac{U_\infty}{\delta} \left(\frac{\delta}{L} \right) \left(\frac{L}{\delta} \right) \sim \mu \frac{U_\infty}{L} Re_L^{1/2} \sim \rho U_\infty^2 Re_L^{-1/2}$$

$$\text{So: } \boxed{\tau \sim \rho U_\infty^2 Re_L^{-1/2}}$$

Defining a dimensionless skin friction coefficient as: C_f

$$C_f = \frac{\tau}{\frac{1}{2} \rho U_\infty^2} \Rightarrow \boxed{C_f \sim Re_L^{-1/2}}$$

We shall see later how well these scaling solutions hold up to more exact results.

Now we can look at the heat transfer aspect of the problem:

$$\underbrace{U \frac{\partial T}{\partial x}}_{\text{Convection}} + \underbrace{V \frac{\partial T}{\partial y}}_{\text{Convection}} = \alpha \frac{\partial^2 T}{\partial y^2}$$

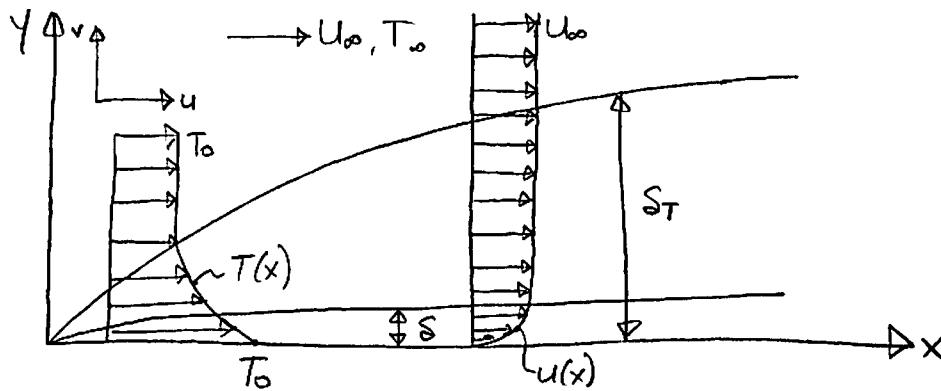
$$\underbrace{U \frac{\Delta T}{L}}_{\text{Convection}} \quad \underbrace{V \frac{\Delta T}{\delta_T}}_{\text{Convection}} \quad \alpha \underbrace{\frac{\Delta T}{\delta_T^2}}_{\text{Conduction}}$$

Here, we are interested in the heat transfer coefficient at the wall, h :

$$h = \frac{-k \frac{\partial T}{\partial y} \Big|_{y=0}}{\Delta T} \sim \frac{k (\Delta T / \delta_T)}{\Delta T} \sim \frac{k}{\delta_T}$$

Now we must be a bit more careful because we are involving flow properties (δ) with heat transfer properties (δ_T). We also can't simply say $U \sim U_\infty$ since the U considered here is within the thermal boundary layer, not the hydrodynamic.

Thick Thermal Boundary Layer ($\delta_T \gg \delta$)



In this limit, the thermal boundary layer is much thicker compared to the velocity boundary layer at any x , i.e. $\frac{\delta}{\delta_T} \ll 1$

Looking back at our energy equation:

$$\underbrace{U \frac{\Delta T}{L} + V \frac{\Delta T}{\delta_T}}_{\text{convection}} \sim \underbrace{\alpha \frac{\Delta T}{\delta_T^2}}_{\text{conduction}}$$

For this case, in the thermal b.l., $U \sim U_\infty$
From continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{U_\infty}{L} + \frac{V}{\delta} \Rightarrow V \sim \frac{U_\infty \delta}{L}$$

So now going back to our scaling

$$U \frac{\Delta T}{L} \sim U_\infty \frac{\Delta T}{L}$$

$$V \frac{\Delta T}{\delta_T} \sim \frac{U_\infty \delta \Delta T}{L \delta_T} \sim \frac{U_\infty \Delta T}{L} \left(\frac{\delta}{\delta_T} \right) \ll 1 \quad (\text{This term is negligible})$$

Note, when scaling for velocity, always use the b.l. that characterizes the flow in the thermal b.l.

Hence, we can now write:

$$U_\infty \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2} \Rightarrow \boxed{\frac{\delta_T}{L} \sim \Pr^{-1/2} \text{Re}_L^{-1/2}} ; \boxed{\Pr = \frac{V}{\alpha}} \quad \text{Prandtl \#}$$

Re-writing this as:

$$\rho_{e_L} = \Pr \cdot Re_L = \text{Peclet Number} = U_0 L / \alpha$$

$= \frac{\text{Rate of thermal advection}}{\text{Rate of thermal diffusion}}$

$$\frac{\delta_T}{L} \sim \rho_{e_L}^{-1/2}$$

Interestingly, we can see that:

$$\frac{s}{L} \sim Re_L^{-1/2} \quad (\text{equation 11}) \quad \text{and} \quad \frac{\delta_T}{L} \sim \rho_{e_L}^{-1/2} \sim \Pr^{-1/2} Re_L^{-1/2}$$

$$\frac{\delta_T}{K} \cdot \frac{L}{s} = \frac{\delta_T}{s} \sim \frac{\Pr^{-1/2} Re_L^{-1/2}}{Re_L^{-1/2}} \sim \Pr^{-1/2}$$

$$\therefore \frac{\delta_T}{s} \sim \Pr^{-1/2} \gg 1 ; \quad \Pr = \frac{U}{\alpha}$$

Therefore, our first assumption of $\frac{\delta_T}{s} \gg 1$ is only valid if $\Pr^{1/2} \ll 1$. This is true for liquid metals ($\Pr \approx 0.0001$).

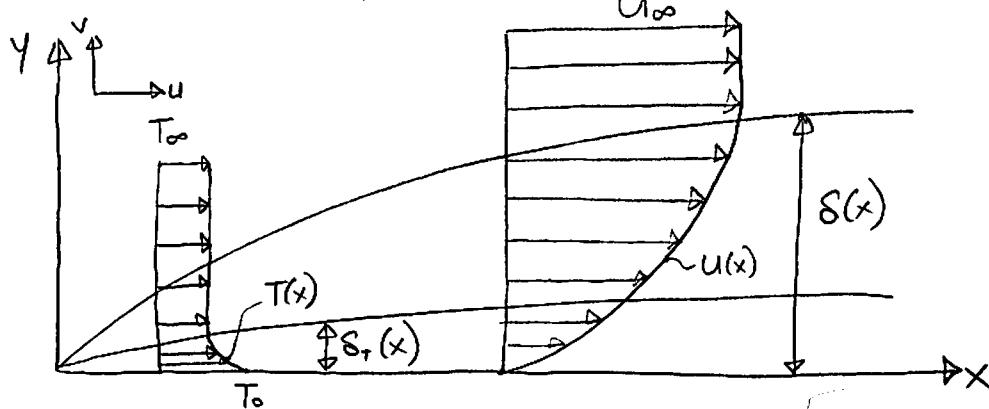
Now we can solve for heat transfer:

$$h \sim \frac{k}{\delta_T} \sim \frac{k}{L} \Pr^{1/2} Re_L^{1/2} \quad \text{for } \Pr \ll 1,$$

Or re-writing this in terms of Nusselt number:

$$Nu = \frac{hL}{k} \sim \Pr^{1/2} Re_L^{1/2}$$

Thin Thermal Boundary Layer ($\delta \approx \delta_T$ or $\delta_T \ll \delta$)



In this limit, the thermal boundary layer is much thinner than the hydrodynamic boundary layer.

Here it gets a bit tricky. When scaling U in the energy equation, we must use the U in the thermal boundary layer, which in this case is not U_∞ !

$$U \neq U_\infty$$

$$U \sim U_\infty \left(\frac{\delta_T}{\delta} \right) \quad (\text{Linear approximation } \Rightarrow \text{ok for scaling})$$

Now we can deal with V :

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{U}{L} + \frac{V}{\delta_T} = 0 \quad \Rightarrow \text{Note here I used } \delta_T \text{ instead of } \delta. \text{ Again, this is due to } \delta_T \text{ being well within } \delta \text{ and the length scale being pertinent to heat trans.}$$

$$V \sim U \left(\frac{\delta_T}{L} \right) \sim U_\infty \frac{\delta_T^2}{SL}$$

Now we can check our convective terms again to see the dominant ones

$$\left. \begin{aligned} U \frac{\Delta T}{L} &\sim U_\infty \frac{\delta_T}{\delta} \cdot \frac{\Delta T}{L} \\ V \frac{\Delta T}{\delta_T} &\sim U_\infty \frac{\delta_T^2}{SL} \cdot \frac{\Delta T}{\delta_T} \sim U_\infty \frac{\delta_T}{\delta} \frac{\Delta T}{L} \end{aligned} \right\} \text{Both are important & must be kept}$$

Back into our energy equation:
convection \sim conduction

$$U_\infty \frac{s_T \Delta T}{L} \sim \alpha \frac{\Delta T}{s_T^2}$$

We know from our hydrodynamic solution that: $\frac{s}{L} \sim Re_L^{-1/2}$
back substituting; (inertia \sim friction)

$$U_\infty \frac{s_T}{L Re_L^{-1/2}} \cdot \frac{1}{L} \sim \alpha \frac{1}{s_T^2} ; \quad Pr = \frac{V}{\alpha} \Rightarrow \alpha = \frac{V}{Pr}$$

$$U_\infty \frac{s_T^3}{L^2 Re_L^{-1/2}} \sim \frac{V}{Pr}$$

$$\frac{s_T^3}{L^3} \sim \frac{V Re_L^{-1/2}}{Pr \cdot L U_\infty} \sim \underbrace{\left(\frac{V}{U_\infty L} \right)}_{Re_L^{-1/2}} Pr^{-1} Re_L^{-1/2}$$

$$\frac{s_T}{L} \sim \left(Pr^{-1} Re_L^{-1/2} \right)^{1/3}$$

$$\boxed{\frac{s_T}{L} \sim Pr^{-1/3} Re_L^{-1/2}} \quad \text{for } \frac{s_I}{s} \ll 1.$$

This means that:

$$\frac{s_T}{s} = \left(\frac{s_T}{L} \right) \cdot \left(\frac{L}{s} \right) \sim Pr^{-1/3} Re_L^{-1/2} \cdot Re_L^{+1/2}$$

$$\boxed{\frac{s_T}{s} \sim Pr^{-1/3} \ll 1} \Rightarrow \text{This is valid for fluids with } Pr^{1/3} \gg 1.$$

These include highly viscous fluids such as oils, honey, etc...
Solving for h & Nu :

$$\boxed{h \sim \frac{h}{s_T} \sim \frac{k}{L} Pr^{1/3} Re_L^{1/2}} \quad (Pr \gg 1)$$

$$\boxed{Nu = \frac{hL}{k} \sim Pr^{1/3} Re_L^{1/2}} \quad (Pr \gg 1)$$

Note we could have done the scaling in terms of a timescale analysis:

$$\text{Conduction} \Rightarrow \tau_{ST} \sim \frac{s_T^2}{\alpha} \quad (\text{from } 0 \sim \sqrt{\alpha t})$$

$$\text{Convection} \Rightarrow \tau_L \sim \frac{L}{U_\infty} \quad (\text{for } Pr < 1) \quad \text{Longitudinal speed} \sim U_\infty$$

$$\tau_L \sim \frac{L}{\frac{s_T}{S} U_\infty} \quad (\text{for } Pr > 1) \quad \text{Longitudinal speed} \sim \frac{s_T}{S} U_\infty$$

Scaling our timescales with each other:

Pr < 1:

$$\tau_{ST} \sim \tau_L \Rightarrow \frac{s_T^2}{\alpha} \sim \frac{L}{U_\infty} \Rightarrow Pe = \frac{U_\infty L}{\alpha}$$

$s_T \sim \sqrt{\frac{\alpha L}{U_\infty}} \sim L \cdot Pe^{-1/2}$

\Rightarrow Same as before.

Pr > 1:

$$\tau_{ST} \sim \tau_L \Rightarrow \frac{s_T^2}{\alpha} \sim \frac{L}{\frac{s_T}{S} U_\infty}$$

$$s_T^3 \sim \frac{SL\alpha}{U_\infty} \Rightarrow S \sim L \cdot Re_L^{-1/2}$$

$$s_T^3 \sim \frac{L^2 \alpha Re_L^{-1/2}}{U_\infty} \sim \frac{L^2 D}{U_\infty \alpha} \cdot Re_L^{-1/2}$$

$s_T \sim L \cdot \alpha^{-1/3} Re_L^{-1/2}$

\Rightarrow Same as before.

Meaning of Reynolds Number

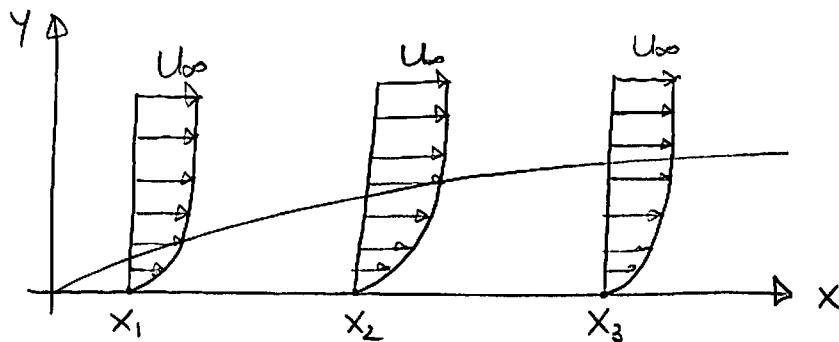
For external flows, $Re_L = U_\infty L / v$. For all of your lives you've been told that $Re_L = \text{Inertia/Friction}$, but we just said that for laminar b.l. Inertia \sim Friction, up to $10^5 = Re_L$. The way to interpret Reynolds number in b.l. flow is geometric:

$Re_L^{1/2} = \frac{L}{S} = \frac{\text{wall length}}{\text{b.l. thickness}}$

\Rightarrow sometimes called "slenderness ratio"

Similarity Solutions

The basic idea to these solutions is the observation that from one location x to another, the U & T profiles look similar. For example:



For all three locations, we know that $U(x=0) = 0$
 $U(x \rightarrow \infty) = U_{\infty}$

From this, we can argue that:

$$\frac{U}{U_{\infty}} = f(\eta) \Rightarrow \eta = \text{similarity variable.}$$

We can intuitively say that $\eta \sim y$, and $\eta = f(x)$.
 From our previous scaling analysis, we can deduce:

$$\eta \sim \frac{y}{\delta(x)} \sim \frac{y}{x R_{ex}^{-1/2}}$$

So we can assume:

$$\boxed{\eta = \frac{y}{x} R_{ex}^{1/2} = y \sqrt{\frac{U_{\infty}}{U x}}} \Rightarrow \text{Similarity Variable.}$$

We also assume that:

$$\boxed{\frac{U}{U_{\infty}} = f(\eta)}$$

Our equations to work with are:

$$U \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} = U \frac{\partial^2 U}{\partial y^2}$$

$$\frac{\partial U}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Back substituting and solving for individual terms:

$$\frac{\partial u}{\partial x} = U_{\infty} \frac{\partial f''}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = U_{\infty} f'' \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial x} = \frac{2}{2x} \left(y \sqrt{\frac{U_{\infty}}{V}} \cdot x^{-1/2} \right) = y \sqrt{\frac{U_{\infty}}{V}} \cdot -\frac{1}{2} x^{-3/2} = -\frac{1}{2x} \eta$$

$$\frac{\partial u}{\partial x} = -U_{\infty} f'' \frac{1}{2x} \eta$$

Now we can use continuity to solve for v.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = U_{\infty} f'' \frac{1}{2x} \eta$$

$$\frac{v}{U_{\infty}} = \frac{1}{2x} \int_0^y f'' \eta dy = \underbrace{\frac{1}{2x} \sqrt{\frac{xU}{U_{\infty}}} \int_0^{\eta} f'' \eta d\eta}_{\frac{1}{2} \cdot Re_x^{-1/2}} \quad (\text{since } dy = \sqrt{\frac{xU}{U_{\infty}}} d\eta)$$

Now we can solve the above integral

$$\int_0^{\eta} \eta f'' d\eta = uv - \int v du \quad (\text{IBP: } \int u dv = uv - \int v du)$$

$$\text{here } u = \eta, \quad v = f', \quad dv = f'' d\eta$$

$$\begin{aligned} \int_0^{\eta} \eta f'' d\eta &= \eta f' - \int_0^{\eta} f' d\eta \\ &= \eta f' - f \end{aligned}$$

$$\text{So: } \frac{v}{U_{\infty}} = \frac{1}{2} Re_x^{-1/2} (\eta f' - f)$$

Now we can solve for our last term:

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{2y} \left(\frac{\partial u}{\partial y} \right) = \frac{2}{2y} \left(U_{\infty} \frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) = \frac{2}{2y} \left(U_{\infty} f'' \frac{Re_x^{1/2}}{x} \right)$$

$$= U_\infty \frac{Rex^{1/2}}{x} \frac{\partial}{\partial y} (f'') = \frac{U_\infty Rex^{-1/2}}{x} \frac{\partial f''}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty Rex}{x^2} f'''$$

Back substituting all of our terms into our momentum eqn.

$$U_\infty f' \left(-U_\infty f'' \frac{1}{2x} \eta \right) + U_\infty \frac{1}{2} Rex^{-1/2} (\eta f' - f) \left(U_\infty f'' \frac{Rex^{1/2}}{x} \right)$$

$$= U \frac{U_\infty Rex}{X^2} f'''$$

$$\cancel{-f' f'' U_\infty \frac{\eta}{2}} + \frac{1}{2} Rex^{-1/2} \eta f' U_\infty f'' Rex^{1/2} - \frac{1}{2} Rex^{-1/2} f U_\infty f'' Rex^{1/2}$$

$$- U \frac{Rex}{X} f''' = 0$$

$$-\frac{1}{2} f U_\infty f'' - U \frac{Rex}{X} f''' = 0$$

We know $Rex = \frac{U_\infty X}{V}$, back substituting

$$\frac{1}{2} f U_\infty f'' + V \frac{U_\infty X}{X V} f''' = 0$$

$$\boxed{\frac{1}{2} f f'' + f''' = 0}$$

$$\boxed{f'(\eta) = \frac{U}{U_\infty}}$$

$$\boxed{\eta = Y \sqrt{\frac{U_\infty}{UX}}}$$

Now we've converted our PDE into an ODE

Converting our boundary conditions:

$$U(y=0) = 0 \Rightarrow \eta(y=0) = 0 ; f'(0) = 0$$

$$U(y \rightarrow \infty) = U_\infty \Rightarrow \eta(y \rightarrow \infty) = \infty ; f'(\eta \rightarrow \infty) = 1$$

$$V(y=0) = 0 \Rightarrow \eta(y=0) = 0 ; V = U_\infty \frac{1}{2} Rex^{-1/2} (\eta(0) \cdot f'(0) - f(0))$$

$$f'(0) = 0$$

To solve, we can assume an infinite series solution:

$$f = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + \dots +$$

$$f' = a_1 + 2a_2 \eta + 3a_3 \eta^2 + \dots +$$

$$f'' = 2a_2 + 6a_3 \eta + \dots +$$

$$f''' = 6a_3 + \dots +$$

Back substituting and solving for our coefficients (matrix form)

$$\underbrace{(\dots)}_0 \eta^0 + \underbrace{(\dots)}_0 \eta^1 + \underbrace{(\dots)}_0 \eta^2 + \underbrace{(\dots)}_0 \eta^3 + \dots + \underbrace{(\dots)}_0 \eta^n = 0$$

We can obtain a recursion formula relating our constants

$$f = \frac{a_2 \eta^2}{2!} - \frac{a_2 \eta^5}{2 \cdot 5!} + \frac{11}{4} \frac{a_3^3 \eta^8}{8!} + \dots$$

\Rightarrow Blasius Solution (1911)
Prandtl's PhD Student

$$a_2 = 0.332 \quad \Rightarrow \text{since } f(0) = f'(0) = 0 \Rightarrow a_1 = a_0 = 0$$

We know that $f'(\eta) = \frac{U}{U_\infty}$

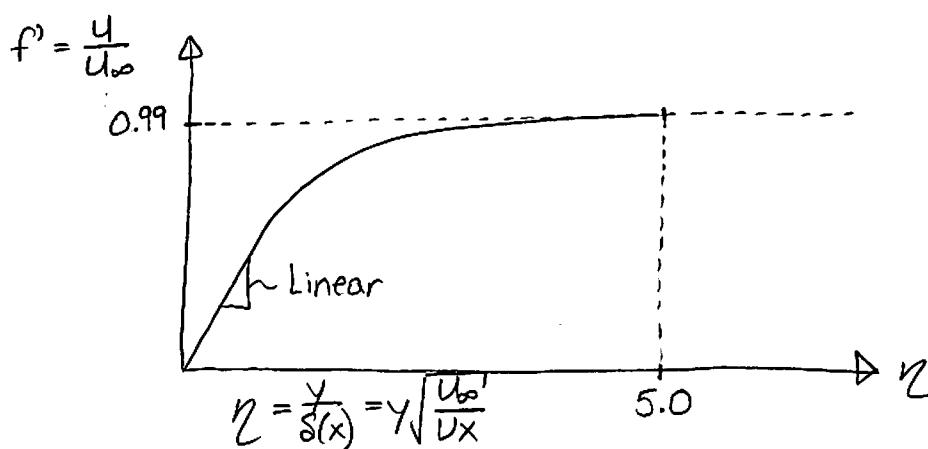
$$f'(\eta) = \frac{2a_2 \eta}{2} + \frac{5 \cdot a_2 \eta^4}{2 \cdot 120} + \left(\frac{11}{4}\right) \frac{8a_3^3 \eta^7}{40320} + \dots$$

$$\eta = \sqrt{\frac{U_\infty}{Ux}}$$

Looking at only our first term: $\eta \rightarrow 0$ (higher order terms drop)

$$f'(\eta) = \frac{U}{U_\infty} = 0.332 \sqrt{\frac{U_\infty}{Ux}} \quad \text{or} \quad U(x,y) = 0.332 \sqrt{\frac{U_\infty^3}{Ux}}$$

Note we can solve numerically and plot our non-dimensional result. Also, the above $U(x,y)$ result is only valid near the wall where $\eta \ll 1$, so h.o.t. drop out.



Solving for when $f' = 0.99$ (when $u = 0.99 U_\infty = b.l. \text{ thickness}$)

$$5.0 = S \sqrt{\frac{U_\infty}{Ux}} = \frac{S}{x} Re_x^{1/2}$$

$$S = \frac{5x}{\sqrt{Re_x}}$$

\Rightarrow Hydrodynamic boundary layer thickness on a flat plate in laminar flow.

Now we can use this super usefull information to do some calculations:

$$C(x) = U \left. \frac{\partial u}{\partial y} \right|_{y=0} \Rightarrow \text{we know } f' = \underbrace{0.332 y \sqrt{\frac{U_\infty}{Ux}}}_{a_2} = \frac{u}{U_\infty} \text{ at } \eta \rightarrow 0$$

$$u = 0.332 U_\infty y \sqrt{\frac{U_\infty}{Ux}} \Rightarrow \text{only the first term from our solution}$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = 0.332 U_\infty \sqrt{\frac{U_\infty}{Ux}}$$

since $\eta \ll 1$, and higher order terms drop out.

$$C(x) = 0.332 U_\infty U \sqrt{\frac{U_\infty \rho}{Ux}} = 0.332 U_\infty \sqrt{\frac{U U_\infty \rho}{x}} \cdot \left(\frac{\rho U_\infty}{\rho U_\infty} \right) \left(\frac{2}{2} \right)$$

$$= 0.664 \cdot \frac{1}{2} \cdot \rho U_\infty^2 \sqrt{\frac{U U_\infty \rho}{\rho^2 U_\infty^2 x}} = 0.664 \frac{1}{2} \cdot \rho U_\infty^2 Re_x^{-1/2}$$

$$\boxed{\frac{C(x)}{\frac{1}{2} \rho U_\infty^2} = 0.664 \cdot Re_x^{-1/2} = C_{f,x}} \Rightarrow \text{Skin friction coefficient}$$

Typically, we want the averaged friction coefficient:

$$\bar{C} = \frac{1}{L} \int_0^L C(x) dx$$

Noting that $C(x) = C x^{-1/2}$, $C = \frac{1}{2} \rho U_\infty^2 \cdot (0.664) U_\infty^{1/2} L^{-1/2}$

$$\bar{C} = C \cdot \frac{1}{L} \int_0^L \frac{dx}{x^{1/2}} = \frac{2C}{L^{1/2}}$$

$$\boxed{\bar{C} = 0.664 \rho U_\infty^2 \cdot Re_L^{-1/2}} \Rightarrow \text{Wall averaged shear}$$

$$\boxed{\bar{C}_f = \frac{\bar{C}}{\frac{1}{2} \rho U_\infty^2} = \frac{1.328}{Re_L^{1/2}}} \Rightarrow \text{Wall averaged skin friction coefficient.}$$

Note on page 41 of the notes, we solved from similarity:

$$\left. \begin{array}{l} \bar{C} \sim \rho U_\infty^2 Re_L^{-1/2} \\ \bar{C}_f \sim Re_L^{-1/2} \end{array} \right\} \begin{array}{l} \text{Pretty good when considering the ease} \\ \text{in which we got these.} \end{array}$$

Example] Calculate the boundary layer thickness at the windshield of a car moving at 70 mph.

$$U_\infty = 70 \text{ mph} (31.11 \text{ m/s})$$

$$\nu_{air} = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$$

$L = 1 \text{ m}$ (length from the front bumper to the windshield)

$$Re_L = \frac{V_\infty L}{\nu} = \frac{(31.11 \text{ m/s})(1 \text{ m})}{(1.5 \times 10^{-5} \text{ m/s})} \approx 2.07 \times 10^6$$

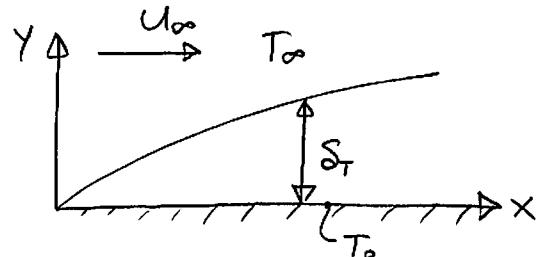
$$S = \frac{5L}{\sqrt{Re_L}} = \frac{5 \text{ m}}{\sqrt{2.07 \times 10^6}} = 0.0035 \text{ m} = 3.5 \text{ mm}$$

$$\boxed{S = 3.5 \text{ mm}} \Rightarrow \text{Very thin!}$$

No wonder Prandtl had a hard time seeing it!

Heat Transfer (Similarity Solution)

$$h = \frac{q''|_{y=0}}{\Delta T} = \frac{q''|_{y=0}}{T_0 - T_\infty} = ?$$



Looking at our energy equation:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$\begin{aligned} T(y=0) &= T_0 \\ T(y \rightarrow \infty) &= T_\infty \end{aligned}$$

Non dimensionalizing: $\Theta = \frac{T - T_0}{T_\infty - T_0}$

$$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \Leftrightarrow U \frac{\partial \bar{u}}{\partial x} + V \frac{\partial \bar{u}}{\partial y} = V \frac{\partial^2 \bar{u}}{\partial y^2}$$

B.l. momentum equation

$$\Theta(y=0) = 0$$

$$\Theta(y \rightarrow \infty) = 1$$

$$\left. \frac{\partial \Theta}{\partial y} \right|_{y \rightarrow \infty} = 0$$

Noting that a useful connection between the energy & hydrodynamic equations is $Pr = V/\alpha$

If $Pr = 1$, $\alpha = V$ and $\Theta = \bar{u}$ and $\delta = \delta_T$

Usually however, $Pr \neq 1$, so we have to solve for Θ

Assuming an identical similarity variable as before

$$\eta = y \sqrt{\frac{U_\infty}{xV}}$$

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \Theta}{\partial \eta} \cdot \left(-\frac{y}{2} \sqrt{\frac{U_\infty}{Vx^3}} \right)$$

$$\frac{\partial \Theta}{\partial y} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\begin{aligned}\frac{\partial^2 \Theta}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial \Theta}{\partial \eta} \right) = \frac{\partial}{\partial y} \left(\frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial \Theta}{\partial \eta} \right) \cdot \left(\frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \Theta}{\partial \eta} \right) \\ &= \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \left(\frac{\partial \Theta}{\partial \eta} \right) \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 \Theta}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2\end{aligned}$$

Back substituting into our energy equation:

$$U \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + V \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \propto \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y} \right)^2 \quad ①$$

To solve, we need to use our hydrodynamic b.l. definitions:
On page 48 of notes, we already solved that:

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2x} \cdot \eta$$

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{xV}}$$

$$U = U_\infty f' \quad (\text{Definition, we defined } f' = \frac{U}{U_\infty})$$

$$V = U_\infty \cdot \frac{1}{2x} \sqrt{\frac{XV}{U_\infty}} (nf' - f) \quad (\text{From continuity})$$

Back substituting into equation ①

$$\begin{aligned}-U_\infty f' \underbrace{\frac{\partial \Theta}{\partial \eta} \cdot \frac{y}{2x} \sqrt{\frac{U_\infty}{xV}}}_{\frac{n}{2}} + \frac{U_\infty}{2x} \sqrt{\frac{XV}{U_\infty}} (nf' - f) \frac{\partial \Theta}{\partial \eta} \sqrt{\frac{U_\infty}{xV}} \\ = \propto \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \frac{U_\infty}{XV}\end{aligned}$$

$$\begin{aligned}-f' \frac{\partial \Theta}{\partial \eta} \cdot \frac{n}{2} + f' \frac{\partial \Theta}{\partial \eta} \cdot \frac{n}{2} \cdot \sqrt{\frac{XV}{U_\infty}} \cdot \sqrt{\frac{U_\infty}{XV}} - \frac{1}{2} f \frac{\partial \Theta}{\partial \eta} \sqrt{\frac{XV}{U_\infty}} \cdot \sqrt{\frac{U_\infty}{XV}} \\ = \propto \frac{\partial^2 \Theta}{\partial \eta^2} \frac{1}{V}\end{aligned}$$

The first two terms cancel and we are left with:

$$\frac{\partial^2 \theta}{\partial \eta^2} + \underbrace{\frac{1}{2} \Pr}_{f} \cdot \frac{\partial \theta}{\partial \eta} = 0$$

$$\boxed{\theta'' + \frac{1}{2} \Pr \cdot f \cdot \theta' = 0} \quad (2)$$

We've turned our energy PDE into an ODE

To solve we can integrate but here we can use a trick:
Usually most books do the following:

$$\frac{d\theta'}{d\eta} + \frac{1}{2} \Pr \cdot f \cdot \theta' = 0$$

$$\frac{d\theta'}{\theta'} + \frac{\Pr \cdot f}{2} d\eta = 0$$

$$\theta = C_1 \int_0^\eta \left[\exp \left(-\frac{\Pr}{2} \int_0^\zeta f d\zeta \right) \right] d\zeta + C_2$$

Using our b.c.'s $\Rightarrow \theta = 0$ at $\eta = 0 \Rightarrow C_2 = 0$

$$\theta = 1 \text{ at } \eta \rightarrow \infty \Rightarrow C_1 = \frac{1}{\int_0^\infty \left[\exp \left(-\frac{\Pr}{2} \int_0^\zeta f d\zeta \right) \right] d\zeta}$$

So our cumbersome solution is:

$$\boxed{\theta(\eta) = \frac{\int_0^\eta \left[\exp \left(-\frac{\Pr}{2} \int_0^\zeta f d\zeta \right) \right] d\zeta}{\int_0^\infty \left[\exp \left(-\frac{\Pr}{2} \int_0^\zeta f d\zeta \right) \right] d\zeta}} \quad (3)$$

$\Rightarrow f(\eta)$ is tabulated from the Blazius solution, so we can solve this numerically.

However, there is an easier way to solve this:
Let:

$$f(\eta^*) = \Pr^{2/3} \cdot f(\eta) \quad (4) \quad \text{and} \quad \eta^* = \eta \cdot \Pr^{1/3} \quad (5)$$

Back substitute (4) and (5) into (2)

$$\Theta'' + \frac{1}{2} f(\eta^*) Pr^{1/3} \frac{\partial \Theta}{\partial \eta} = 0$$

From our definition (eq. 5) $\Rightarrow \eta^* = \eta Pr^{1/3}$
 $d\eta^* = Pr^{1/3} d\eta$

$$\Theta'' + \frac{1}{2} f(\eta^*) Pr^{2/3} \underbrace{\frac{\partial \Theta}{\partial \eta}}_{\partial \eta^*} = 0 \quad (6)$$

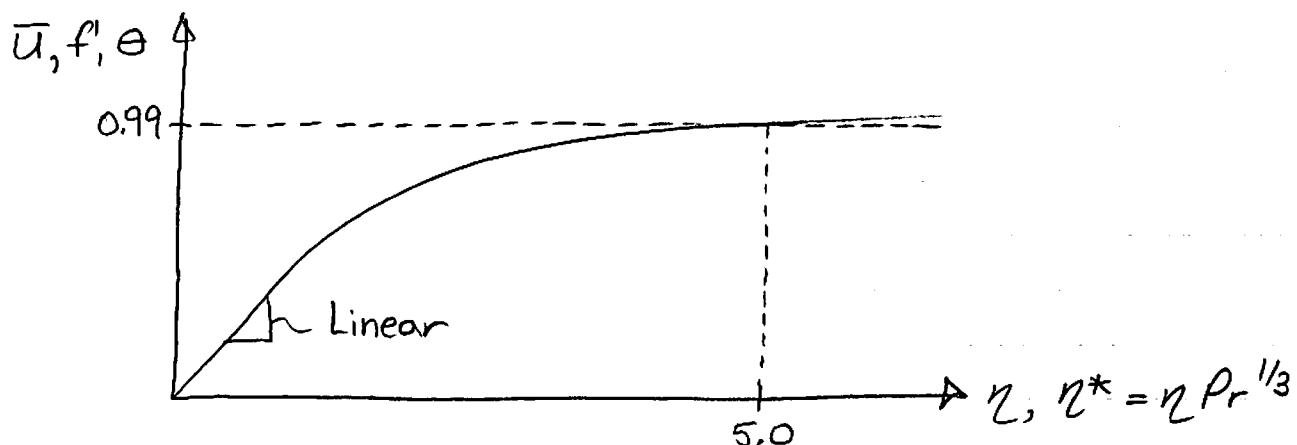
Multiply ⑥ through by $Pr^{-2/3}$

$$\underbrace{\frac{1}{Pr^{2/3}} \left(\frac{\partial^2 \Theta}{\partial \eta^*} + \frac{1}{2} f(\eta^*) Pr^{2/3} \frac{\partial \Theta}{\partial \eta^*} \right)}_{2\eta^{*2}} = 0$$

Now our PDE becomes identical to the Blasius ODE. We already have the solution.

$$\Theta'' + \frac{1}{2} f(\eta^*) \Theta' = 0 \Rightarrow \Theta(\eta^*) \Leftrightarrow \bar{U}(\eta) = f'$$

Our b.c.'s are: $\Theta(\eta^* = 0) = 0$ $\Theta(\eta^* \rightarrow \infty) = 1$ $\left. \begin{array}{l} \Theta = \frac{T - T_0}{T_\infty - T_0} \\ \bar{U} = f' \end{array} \right\}$



η = hydrodynamic b.l.
 η^* = thermal b.l.

For the hydrodynamic boundary layer, we had:

$$\delta \sqrt{\frac{U_\infty}{xU}} = \eta(y=\delta) = 5.0 \Rightarrow \boxed{\delta(x) = \frac{5x}{\sqrt{Re_x}}}$$

Now for the thermal boundary layer, our solution is similar

$$Pr^{1/3} \delta_T \sqrt{\frac{U_\infty}{xU}} = \eta^*(y=\delta_T) = 5.0 \Rightarrow \boxed{\delta_T(x) = \frac{5x}{Re_x^{1/2} Pr^{1/3}}}$$

Taking the ratio of our two boundary layer thicknesses:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}} \Rightarrow \text{Kind of intuitive since } Pr = \frac{U}{\alpha} = \frac{\text{hydrodynamics}}{\text{energy}}$$

Note before on page ④ of notes, we found with scaling that:

$$\delta_T \sim L Pr^{-1/3} Re_L^{-1/2} \quad \left. \right\} \text{Here we have } \delta_T(L) = 5L Re_L^{1/2} Pr^{1/3}$$

Now we can solve for our heat transfer (note we know: $h \sim \frac{k}{\delta_T}$)

$$q'' \Big|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} \Rightarrow \Theta = \frac{T - T_0}{T_\infty - T_0} ; \quad \frac{\partial \Theta}{\partial y} = \frac{\partial T}{\partial y} \Big|_{y=0}$$

$$\eta^* = \eta Pr^{1/3} = y \left(\frac{U_\infty}{xU} \right)^{1/2} Pr^{1/3}$$

$$\frac{\partial \eta^*}{\partial y} = \frac{\partial y}{\partial x} \left(\frac{U_\infty}{xU} \right)^{1/2} Pr^{1/3}$$

$$q'' \Big|_{y=0} = -k (T_\infty - T_0) \left(\frac{U_\infty}{xU} \right)^{1/2} Pr^{1/3} \frac{\partial \Theta}{\partial \eta^*} \Big|_{\eta^*=0}$$

$$= + \frac{k (T_0 - T_\infty)}{x} \left(\frac{U_\infty x}{U} \right)^{1/2} Pr^{1/3} f''(0)$$

Remember we already know $f \Rightarrow f = \frac{\alpha_2 \eta^2}{2!} - \frac{\alpha_2 \eta^5}{2 \cdot 5!} + \frac{11}{4} \frac{\alpha_3 \eta^8}{8!} \dots$
From page ⑤ of notes

$$f''(0) = \alpha_2 = 0.332$$

$$\left. q'' \right|_{y=0} = \frac{h \Delta T}{x} Re_x^{1/2} Pr^{1/3} \alpha_2$$

$$Nu_x = \frac{hx}{k} = \underbrace{\frac{q''|_{y=0}}{\Delta T}}_h \cdot \frac{x}{k} = \alpha_2 Re_x^{1/2} Pr^{1/3}$$

$$Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}$$

From scaling: $Nu_L \sim Re_L^{1/2} Pr^{1/3}$
 Valid for $Pr > 0.5, T_o = \text{constant}$

Colburn Analogy

We can relate our fluid flow & heat transfer solutions

$$C_{f,x} = \frac{C}{\frac{1}{2} \rho U_\infty^2} = \frac{2\alpha_2}{Re_x^{1/2}} \quad (\text{Skin friction coefficient})$$

$$Nu_x = \alpha_2 Re_x^{1/2} Pr^{1/3}$$

Let's try the following

$$\frac{Nu_x}{Re_x \cdot Pr} = \frac{\alpha_2 Re_x^{1/2} Pr^{1/3}}{Re_x \cdot Pr} = \frac{\alpha_2}{Re_x^{1/2} Pr^{2/3}} = \frac{1}{2} \overbrace{\left(\frac{2\alpha_2}{Re_x^{1/2}} \right)}^{C_{f,x}} \frac{1}{Pr^{2/3}}$$

So what's the point of dividing by Pr ?

Let's expand our definition:

$$\frac{Nu_x}{Re_x \cdot Pr} = \frac{hx}{k} \cdot \frac{\mu}{\rho U_\infty} \cdot \frac{\alpha}{V} = \frac{h}{k} \cdot \frac{\mu}{\rho U_\infty} \cdot \frac{k}{\rho C_p} \cdot \frac{V}{\mu} = \frac{h}{\rho C_p U_\infty}$$

$$\boxed{\frac{Nu_x}{Re_x \cdot Pr} = \frac{h}{\rho C_p U_\infty} = St \equiv \text{Stanton Number}}$$

$$St = \frac{\text{heat transfer to fluid}}{\text{thermal capacity of fluid}} \sim \frac{T_w - T_b}{T_{b,t=2} - T_{b,t=1}}$$

We can now say the following:

$$St \cdot Pr^{2/3} = \frac{C_{fix}}{2} = j_H \equiv \text{Colburn } j\text{-factor}$$

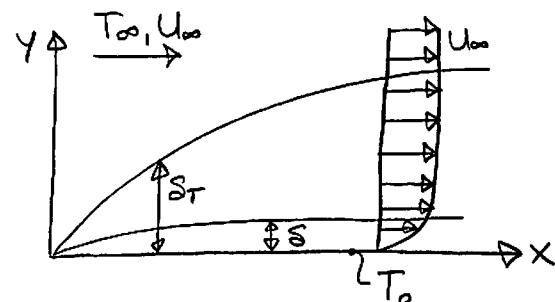
(or Colburn-Chilton analogy)

Very useful for heat transfer analysis as it relates transport properties to one another, (heat, momentum, and mass transfer).

Note the previous solution is valid for $Pr \approx 1$, or $Pr \gg 1$. What if $Pr \ll 1$.

Think about why this changes:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$



If $Pr \ll 1$, $U \propto Re_x^{-1/2}$ since $\delta \ll \delta_T$

Hence $U \sim U_\infty$. Our energy equation becomes:

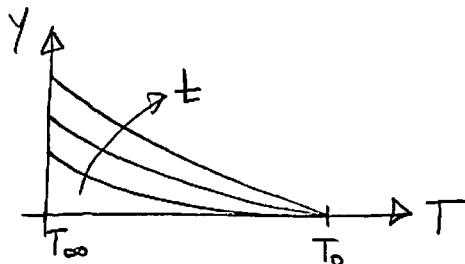
$$U_\infty \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

outside the b.e.

From continuity: $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow U = U_\infty = \text{constant} \Rightarrow V = \text{const.} = 0$

$$U_\infty \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \text{but } x = U_\infty \cdot t \Rightarrow dx = U_\infty dt$$

$$U_\infty \frac{\partial T}{U_\infty dt} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \text{Transient conduction!!!}$$



$$q''_{y=0} = \frac{k \Delta T}{(\pi \alpha t)^{1/2}} \Rightarrow \text{Solved in ME420}$$

Look up in conduction

$$Nu_x = \left(\frac{q''_{y=0}}{\Delta T} \cdot \frac{x}{k} \right) = \frac{x}{(\pi \alpha t)^{1/2}} ; t = \frac{x}{U_\infty}$$

$$Nu_x = \frac{1}{\pi^{1/2}} \cdot Re_x^{1/2} Pr^{1/2} \Rightarrow \begin{aligned} &\text{Pr} < 0.5 \\ &T_0 = \text{constant} \end{aligned}$$

Average Quantities

$$\overline{h} = \underbrace{\frac{\dot{q}''}{\Delta T}}_{T_0 = \text{constant}} \quad \text{or} \quad \overline{h} = \underbrace{\frac{\dot{q}''}{\Delta T}}_{\dot{q}''|_{y=0} = \text{constant}}$$

For $T_0 = \text{constant}$:

$$\overline{h} = \frac{1}{\Delta T} \left[\frac{1}{L} \int_0^L \dot{q}'' dx \right] = \frac{1}{L} \int_0^L h dx$$

For $\dot{q}''|_{y=0} = \text{constant}$:

$$\overline{h} = \frac{\dot{q}''|_{y=0}}{\frac{1}{L} \int_0^L \Delta T dx}$$

The average $Nu_L = \overline{Nu_L} = \frac{\overline{h}L}{k} \neq \frac{1}{L} \int_0^L Nu_x dx$

For a flat plate:

$$\overline{h} = \frac{1}{L} \int_0^L h dx \Rightarrow h = \frac{k}{x} Nu_x$$

$$\overline{h} = \frac{0.332 k \Pr^{1/3}}{L} \sqrt{\frac{U_\infty}{V}} \int_0^L \frac{x^{1/2}}{x} dx \Rightarrow \boxed{\overline{h} = 0.664 Re_L^{1/2} \Pr^{1/3} \cdot \frac{k}{L}}$$

 $\hookrightarrow T_0 = \text{constant}, \Pr > 0.5$

$$\boxed{\overline{Nu_L} = \frac{\overline{h}L}{k} = 0.664 Re_L^{1/2} \Pr^{1/3}} \quad \Pr > 0.5$$

$$\boxed{Nu_L = 1.13 Re_L^{1/2} \Pr^{1/2}} \quad \Pr < 0.5$$

Some Observations and Notes

So far, our results are valid for the following conditions

$$1) Re_x = \frac{\rho U_\infty x}{\mu} \quad \text{or} \quad Re_L = \frac{\rho U_\infty L}{\mu} < 5.0 \times 10^5 \quad (\text{Laminar})$$

$$2) Ma = \frac{U_\infty}{\text{sound speed}} < 0.3 \quad (\text{Incompressible})$$

$$3) Ec \equiv \text{Eckert Number} = \frac{U_\infty^2}{C_p(T_0 - T_\infty)} \ll 1 \Rightarrow \begin{matrix} \text{(Viscous dissipation} \\ \text{heating is negligible)} \end{matrix}$$

4) Evaluate fluid properties at the b.l. film temp.:

$$T_f = \frac{T_0 + T_\infty}{2}$$

$$*5) h_x \sim \frac{1}{x^{1/2}} \Rightarrow \text{As } x \rightarrow 0, h_x \rightarrow \infty$$

The boundary layer model breaks down in the region of $x=0$.

6) So far, we've only dealt with $T_0 = \text{constant}$. We will solve for the $q''_{y=0} = \text{constant}$ case later, with the integral technique.

* Note although the b.l. solution diverges at $x \rightarrow 0$, in real life, h_x is actually higher in this region. Hence it's more beneficial to re-start your b.l. as often as possible to minimize S_T and maximize entrance effects. We will discuss these later in the class.

i.e.



\Rightarrow Good, $h \uparrow$, however $C \uparrow$ as well



\Rightarrow Bad, $h \downarrow$, however $C \downarrow$ as well

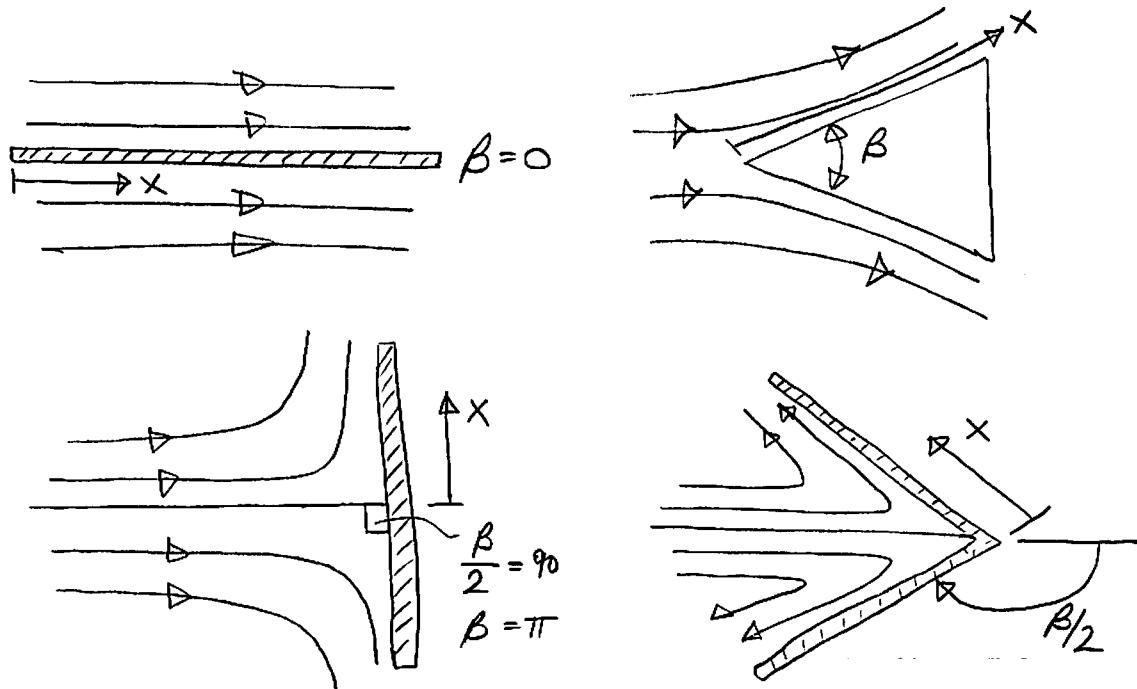
Similarity Solutions for Flow with Longitudinal Pressure Grad.
 So how do we deal with flows which have longitudinal pressure gradients?

It turns out there is a class of similarity solutions that work for potential flow problems, in 2D.

For 2D potential flows:

$$U_\infty(x) = Cx^m \Rightarrow \text{For a derivation of this, visit a potential flow fluids textbook.}$$

$$m = \frac{\beta}{2\pi - \beta} = \frac{x}{U_\infty} \cdot \frac{dU_\infty}{dx}$$



Note for these cases, we cannot assume $\frac{\partial P}{\partial x} = 0$, except for $\beta = 0$. The varying cross section in each flow induces a pressure change in the longitudinal direction.

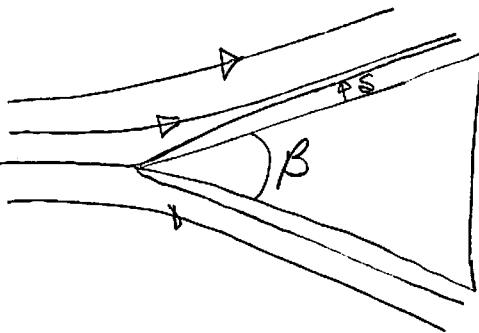
Also note, if $\beta < 0$, this means:



Our b.l. equation (momentum) becomes:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{1}{\rho} \frac{\partial P}{\partial X} + V \frac{\partial^2 U}{\partial Y^2} \quad (1)$$

Taking a streamline at the edge of the boundary layer, or within a b.l. reveals: (note $\frac{\partial P}{\partial Y} \ll \frac{\partial P}{\partial X}$ still)



$$P_{\infty} + \frac{1}{2} \rho U_{\infty}^2 = \text{constant}$$

$$\frac{\partial P_{\infty}}{\partial X} + \rho U_{\infty} \frac{\partial U_{\infty}}{\partial X} = 0$$

$$-\frac{\partial P}{\partial X} = \rho U_{\infty} \frac{\partial U_{\infty}}{\partial X} \quad (2)$$

Back substituting (2) into (1), we obtain:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = U_{\infty} \frac{\partial U_{\infty}}{\partial X} + V \frac{\partial^2 U}{\partial Y^2} \quad (3)$$

$$\text{We know } U_{\infty} = Cx^m \Rightarrow U_{\infty} \frac{\partial U_{\infty}}{\partial X} = Cx^m m C x^{m-1}$$

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{U_{\infty}^2 m}{X} + V \frac{\partial^2 U}{\partial Y^2} \quad (4)$$

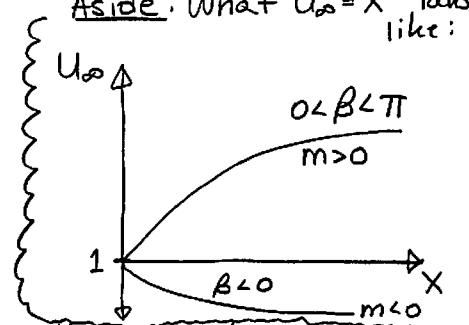
To solve equation (4) we can actually use the identical similarity variable we defined earlier.

$$\zeta = Y \sqrt{\frac{U_{\infty}}{VX}}$$

Instead of using the same procedure as before (i.e. defining $f' = U/U_{\infty}$, & solving for $U, V = f(f', \zeta)$). We can define a streamfunction $\psi(x, y)$:

$$U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X}$$

Aside: What $U_{\infty} = x^m$ looks like:



To check if this works, we can check continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0 \Rightarrow$ Satisfies continuity (Note, only valid for analytic functions)

Now we can do one more thing:

$$f' = \frac{u}{u_\infty} \Rightarrow u = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} = 0$$

$$f' = \frac{1}{u_\infty} \cdot \frac{\partial \psi}{\partial y}$$

$$U_\infty f' = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \Rightarrow \eta = y \sqrt{\frac{U_\infty}{U_\infty U_x}} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{U_x}}$$

$$U_\infty f' \frac{\sqrt{U_x}}{\sqrt{U_\infty}} = \frac{\partial \psi}{\partial \eta}$$

$$\psi = \sqrt{U_\infty U_x} \int f' d\eta \Rightarrow \boxed{\psi = (U_\infty U_x)^{1/2} f} \quad (5)$$

Now we can transform our b.l. equation

$$u \frac{\partial U}{\partial x} + v \frac{\partial U}{\partial y} = \frac{U_\infty^2 m}{x} + v \frac{\partial^2 U}{\partial y^2} \quad (6)$$

Continuity

Back substitute (5) and $u = \frac{\partial \psi}{\partial y}$, $v = -\int \frac{\partial \psi}{\partial x} dy$ into (6)

$$\text{Also, you will need: } \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{2}{\partial x} \left(y \sqrt{\frac{U_\infty}{U_x}} \right)$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{U_x}}$$

$$\frac{\partial \psi}{\partial \eta} = (U_\infty U_x)^{1/2} f' ; \text{ Remember } U_\infty = C x^m$$

Simplifying, we obtain: $\boxed{f''' + \frac{1}{2}(m+1)f f'' + m(1-(f')^2) = 0}$

↳ For full derivation, see pg. 64A-64D

Extra Derivation: Falkner-Skan Momentum Equation

We want to go from PDE to ODE:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{U_\infty^2 m}{X} + V \frac{\partial^2 U}{\partial Y^2} \quad (1)$$

Let's assume our streamfunction formulation:

$$U = \frac{\partial \psi}{\partial Y} ; \quad V = -\frac{\partial \psi}{\partial X} ; \quad \eta = Y \sqrt{\frac{U_\infty}{VX}} ; \quad \psi = (U_\infty VX)^{1/2} f \quad (2)$$

We will need the following quantities to help us:

$$\frac{\partial \eta}{\partial Y} = \frac{\partial}{\partial Y} \left(Y \sqrt{\frac{U_\infty}{VX}} \right) = \sqrt{\frac{U_\infty}{VX}} \Rightarrow \boxed{\frac{\partial \eta}{\partial Y} = \sqrt{\frac{U_\infty}{VX}}} \quad (3)$$

$$\begin{aligned} \frac{\partial \eta}{\partial X} &= \frac{\partial}{\partial X} \left(Y \sqrt{\frac{U_\infty}{VX}} \right) = U_\infty = X^m \\ &= \frac{\partial}{\partial X} \left(Y \sqrt{\frac{X^m}{VX}} \right) = \frac{Y}{\sqrt{V}} \frac{\partial}{\partial X} \sqrt{X^{m-1}} \\ &= \frac{Y}{\sqrt{V}} \frac{1}{2} (m-1) X^{\frac{1}{2}(m-1)-1} \end{aligned}$$

$$\begin{aligned} &= \frac{Y}{\sqrt{V}} \frac{1}{2} \frac{m-1}{X} \sqrt{\frac{X^m}{X}} \\ &= Y \frac{(m-1)}{2X} \sqrt{\frac{U_\infty}{VX}} \end{aligned}$$

$$\boxed{\frac{\partial \eta}{\partial X} = \frac{(m-1)}{2X} \eta} \quad (4)$$

$$\text{Also remember that } \frac{U}{U_\infty} = f' \Rightarrow \boxed{U = U_\infty f'} \quad (5)$$

And from mass conservation, we can solve for V :

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

We know that:

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial}{\partial x}(U_\infty f') = \frac{\partial U_\infty}{\partial x} f' + \frac{\partial f'}{\partial x} U_\infty \\ &= \frac{\partial}{\partial x}(x^m) f' + \frac{\partial f'}{\partial x} \cdot \frac{\partial \eta}{\partial x} \cdot U_\infty \\ \therefore \frac{\partial U}{\partial x} &= \underbrace{m x^{m-1} f'}_{m U_\infty x^{-1} \text{ since } U_\infty = x^m} + U_\infty f'' \frac{(m-1)}{2x} n \quad (6)\end{aligned}$$

So now we can solve for V :

$$\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x} \quad (\text{from continuity})$$

$$\frac{\partial V}{\partial y} = -\frac{U_\infty}{x} \left(m f' + (m-1) f'' n \cdot \frac{1}{2} \right)$$

$$V = \int -\frac{U_\infty}{x} \left(m f' + \frac{(m-1)}{2} f'' n \right) dy$$

$$\text{We know } \eta = y \sqrt{\frac{U_\infty}{Ux}} \Rightarrow dy = d\eta \sqrt{\frac{Ux}{U_\infty}}$$

$$V = -\frac{U_\infty m}{x} \sqrt{\frac{Ux}{U_\infty}} \int f' d\eta - \underbrace{\frac{U_\infty (m-1)}{2x} \sqrt{\frac{Ux}{U_\infty}} \int f'' n d\eta}_{\text{We've solved this before using IBP. Look on pg.}}$$

$$V = -\frac{U_\infty m}{x} \sqrt{\frac{Ux}{U_\infty}} f - \frac{U_\infty (m-1)}{2x} \sqrt{\frac{Ux}{U_\infty}} (n f' - f')$$

Expanding this:

$$V = -\frac{U_\infty m}{x} \sqrt{\frac{Ux}{U_\infty}} f - \frac{U_\infty m}{2x} \sqrt{\frac{Ux}{U_\infty}} (n f' - f) + \frac{U_\infty}{2x} \sqrt{\frac{Ux}{U_\infty}} (n f' - f)$$

One more step of expansion:

$$V = -\frac{U_\infty}{X} \sqrt{\frac{UX}{U_\infty}} \cdot \left[mf + \frac{m\eta}{2} f' - \frac{mf'}{2} - \frac{\eta f''}{2} + \frac{f''}{2} \right]$$

$$\boxed{V = -\frac{U_\infty}{X} \sqrt{\frac{UX}{U_\infty}} \cdot \left[\frac{mf}{2} + \frac{m\eta}{2} f' - \frac{\eta f'}{2} + \frac{f''}{2} \right]} \quad (7)$$

Now it becomes easy. We just have to plug & chug!
Let's solve the viscous term (Right hand side)

$$\begin{aligned} V \frac{\partial^2 U}{\partial y^2} &= V \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} (U) \right) \Rightarrow U = U_\infty f' \\ &= V \frac{\partial}{\partial y} \left(\frac{\partial}{\partial \eta} (U_\infty f') \frac{\partial \eta}{\partial y} \right) \\ &= V \frac{\partial^2}{\partial \eta^2} (U_\infty f') \left(\frac{\partial \eta}{\partial y} \right)^2 \\ &= \nu U_\infty f''' \left(\frac{U_\infty}{\nu X} \right) \end{aligned}$$

$$\therefore \boxed{V \frac{\partial^2 U}{\partial y^2} = \frac{U_\infty^2 f'''}{\nu X}} \quad (8)$$

Let's do the first inertial term:

$$U \frac{\partial U}{\partial x} \Rightarrow U = U_\infty f', \quad \frac{\partial U}{\partial x} \Rightarrow \text{eqn. } (6)$$

$$= U_\infty f' \left(\frac{m U_\infty f'}{X} + U_\infty f'' \left(\frac{m-1}{2X} \right) \eta \right)$$

$$\boxed{U \frac{\partial U}{\partial x} = \frac{U_\infty^2}{X} \left(m(f')^2 + \frac{m}{2} \eta f' f'' - \frac{1}{2} \eta f' f'' \right)} \quad (9)$$

Now for the second inertial term: Eqn.③

$$\sqrt{\frac{\partial u}{\partial y}} \Rightarrow v \Rightarrow \text{eqn. ⑦}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}^{\nearrow} = \frac{\partial (U_{\infty} f)}{\partial \eta} \sqrt{\frac{U_{\infty}}{U_x}}$$

$$= U_{\infty} f'' \sqrt{\frac{U_{\infty}}{U_x}}$$

$$\sqrt{\frac{\partial u}{\partial y}} = -\frac{U_{\infty}}{2x} \sqrt{\frac{U_x}{U_{\infty}}} [mf + mnf' - nf' + f] \cdot U_{\infty} f'' \sqrt{\frac{U_{\infty}}{U_x}}$$

$$\boxed{\sqrt{\frac{\partial u}{\partial y}} = \frac{U_{\infty}^2}{2x} (nf''f' - mff'' - mnf'f'' - ff'')} \quad ⑩$$

Putting ⑧, ⑨, & ⑩ together: $\text{⑧} + \text{⑩} = \text{⑧} + \frac{U_{\infty}^2 m}{x}$
 $U_{\infty} \frac{\partial u}{\partial x} + \sqrt{\frac{\partial u}{\partial y}} = U_{\infty} \frac{\partial^2 u}{\partial y^2} + \frac{U_{\infty}^2 m}{x}$

$$\cancel{\frac{U_{\infty}^2}{x} (m(f')^2 + \frac{m}{2} nf'f'' - \frac{1}{2} nf'f'' + \frac{1}{2} nf'f'' - \frac{m}{2} ff'')}$$

$$- \cancel{\frac{1}{2} mnf'f'' - \frac{1}{2} ff'')} = \frac{U_{\infty}^2}{x} (m + f''')$$

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

$$\boxed{f''' + \frac{1}{2}(m+1)ff'' + m(1 - (f')^2) = 0}$$

→ Falkner-Skan Wedge Flow Momentum Equation.

Now you try the energy eqn!

Note our boundary conditions remain the same:

$$f(0) = 0 ; f'(0) = 0 ; f'(\infty) = 1$$

Note, the solution can be found numerically and is usually tabulated.

Typically for flow problems we need to solve for shear (τ)

$$\begin{aligned} \tau &= \mu \frac{\partial u}{\partial y} \Big|_{y=0} \Rightarrow u = f' U_\infty \xrightarrow{\int \frac{U_\infty}{U_x}} \\ &= \mu U_\infty \frac{\partial f'}{\partial y} \Big|_{y=0} = \mu U_\infty \left[\frac{\partial f'}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right]_{y=0} \end{aligned}$$

Simplifying: (see page 51 for a similar simplification)

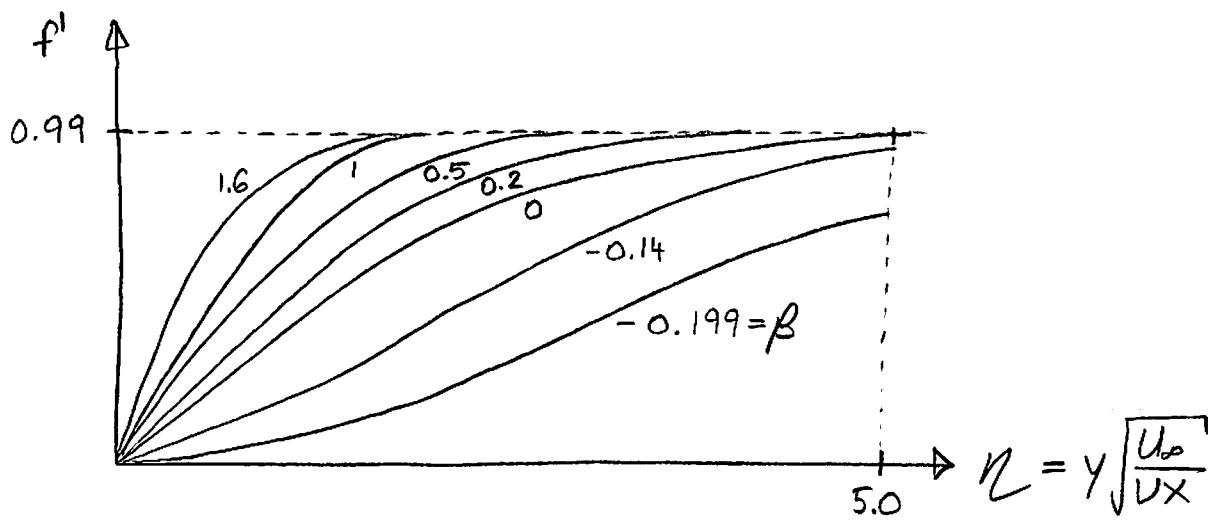
$$C_{f,x} = \frac{\tau}{\frac{1}{2} \rho U_\infty^2} = \frac{2 f''(0)}{Re_x^{1/2}} \Rightarrow \text{Note, } Re_x = \frac{Cx^{m+1}}{U}$$

↳ Don't forget this!

Our tabulated solutions are:

β	m	$f''(0) = \frac{1}{2} C_{f,x} Re_x^{-1/2}$
$2\pi = 6.28$	∞	∞
$\pi = 3.14$	1	1.233
$\pi/2 = 1.57$	$1/3$	0.757
$\pi/5 = 0.627$	$1/9$	0.512
0	0	0.332 \Rightarrow flat plate ($\frac{\partial p_\infty}{\partial x} = 0$)
-0.14	-0.0654	0.164
-0.199	-0.0904	0 (b.l. separation)

If we plot our velocity profile results



From this, we see that for $\beta > 0$, $m > 0$

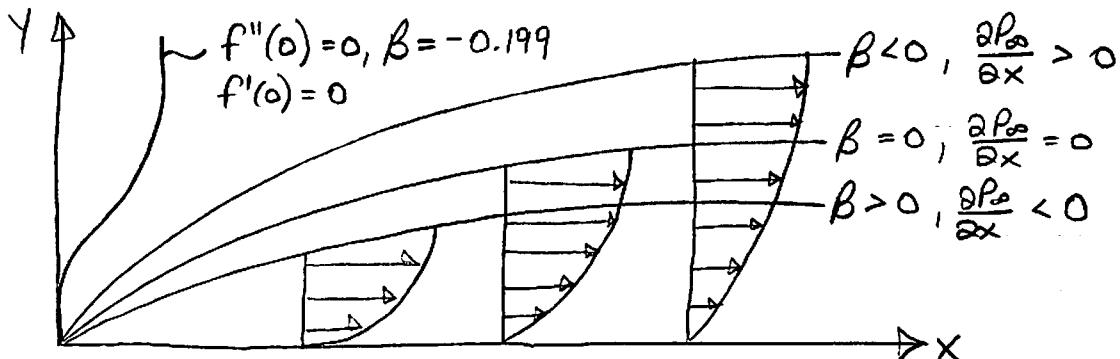
$$\frac{\partial P_\infty}{\partial x} = - \frac{\rho U_\infty^2 m}{x} < 0 \Rightarrow \text{Pressure drops as the flow accelerates}$$

This is a favourable pressure gradient, and our boundary layer gets thinner with larger x -momentum near the wall.

For $\beta < 0$, $m < 0$

$$\frac{\partial P_\infty}{\partial x} = - \frac{\rho U_\infty^2 m}{x} > 0 \Rightarrow \text{Adverse pressure gradient}$$

We see that at $\beta = -0.199$, we have $f''(0) = 0$, so this is called the b.l. separation point



Note, these solutions are called the Fallner-Skan solutions (1931)

Heat Transfer (Falkner - Skan \Rightarrow Wedge Flow)

Our energy equation is exactly the same as before. We can also use the same similarity variables (page 53 of notes)

$$\Theta = \frac{T - T_0}{T_\infty - T_0}$$

$$u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \Rightarrow \eta = y \sqrt{\frac{U_\infty}{Ux}}, f' = \frac{u}{U_\infty}$$

Remember the following:

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{2}{2x} \left(y \sqrt{\frac{Cx^m}{Ux}} \right)$$

$$\frac{\partial \Theta}{\partial y} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \Rightarrow \frac{\partial \eta}{\partial y} = \sqrt{\frac{Cx^m}{Ux}}$$

$$\frac{\partial^2 \Theta}{\partial y^2} = \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y} \right)^2 = \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \frac{Cx^m}{Ux}$$

Back substituting & doing simplification, we obtain:

$$\boxed{\Theta'' + \frac{1}{2} Pr(m+1) f \Theta' = 0} \quad ①$$

This is similar to before except that Pr is replaced with $Pr(m+1)$. Our b.c.'s are identical:

$$\Theta(0) = 0; \quad \Theta(\infty) = 1$$

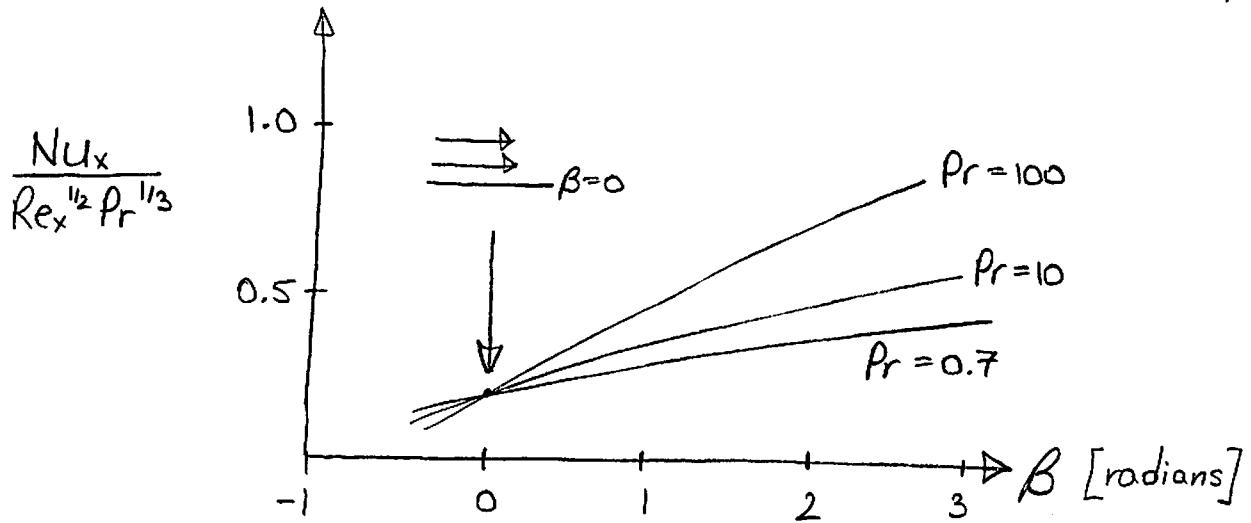
Eckert numerically integrated equation ① & obtained (for $Pr=1$)

β	m	$Nu_x / Re_x^{1/2}$
-0.512	-0.0753	0.272
0	0	0.332
$\pi/5$	$1/9$	0.378
$\pi/2$	$1/3$	0.440
π	1	0.570

$$Nu = \frac{hx}{k}$$

\Rightarrow Table 2.3 of Bejan [pg]

There is a good way to summarize our results graphically



For any given constant Pr , we can see that:

$$\frac{\text{Nu}_x}{\text{Re}_x^{-1/2} \text{Pr}^{1/3}} = \text{constant}$$

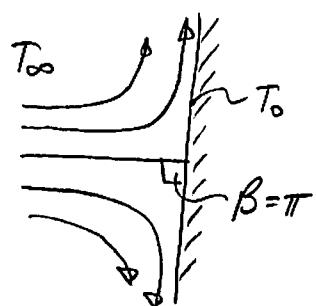
$$\frac{h_x}{k} \left(\frac{x U_\infty}{V} \right)^{-1/2} = \text{constant} \Rightarrow h = \frac{(\text{const.}) k x^{-1/2} U_\infty^{1/2}}{V^{1/2}}$$

But for our problem, $U_\infty = C x^m$

$$h = \frac{(\text{const}) k}{V^{1/2}} x^{(m-1)/2}$$

⇒ heat transfer coefficient (local)

Note a spacial case here, for $m=1$, $\beta=\pi$



$$h = \frac{(\text{const}) k}{V^{1/2}} x^{(1-1)/2} = \frac{(\text{const}) k}{V^{1/2}} = \text{CONSTANT}$$

since $h_x = \text{constant}$, this implies that $S_T = \text{const.}$ for $m=1$. Also, $S = \text{constant}$ for this case.

We can say something about the average heat transfer coeff (\bar{h})

$$\bar{h} = \frac{1}{x} \int_0^x h dx$$

$$\bar{h} = \frac{1}{x} \int_0^x \frac{Ck}{V^{1/2}} \times x^{(m-1)/2} dx = \frac{2}{m+1} \frac{Ck}{V^{1/2}} \times x^{(m-1)/2}$$

$$\boxed{\frac{h}{\bar{h}} = \frac{2}{m+1}}$$

$\Rightarrow h$ is the local heat transfer coefficient ($h(x)$)

Note in real experiments, for $\text{Pr} \approx 1$ ($0.5 < \text{Pr} < 10$) and $m=1$:

$$\boxed{Nu_x = 0.57 Re_x^{1/2} \text{Pr}^{0.4}} \Rightarrow \text{Jet impinging on a wall (2-D)}$$

note: $Re_x = \frac{U_\infty x}{V}$

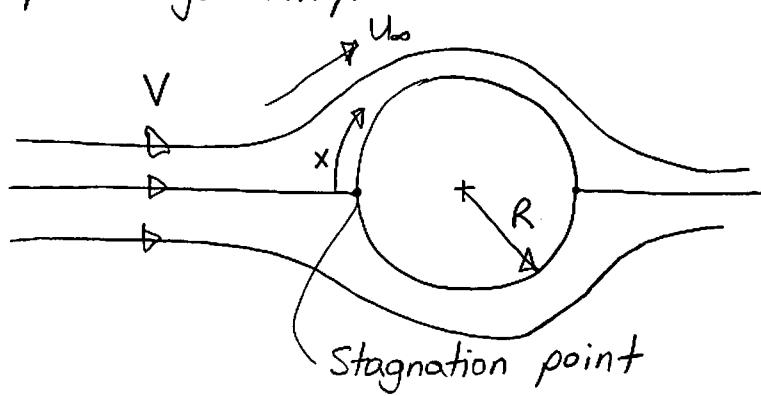
For a 3-D stagnation point:

$$\boxed{Nu_x = 0.76 Re_x^{1/2} \text{Pr}^{0.4}} \Rightarrow \text{Jet impinging on a wall (3-D)}$$

Makes sense that $h_{x,3D} > h_{x,2D}$ since we create a third dimension to dissipate energy & momentum. This is similar to turbulence!

Cylinders and Spheres

To use the above jet impingement solutions ($Nu_x Re_x^{-1/2} = \text{const}$) it is useful to know the variation of U_∞ near the stagnation point: Most solutions can be approximated by a cylinder or sphere geometry.



$$U_\infty = \frac{2Vx}{R} \text{ for small } \frac{x}{R}$$

\hookrightarrow For a cylinder

We know our solution for a 2-D stagnation point is: ($m=1$)

$$Nu_x = 0.57 Re_x^{1/2} Pr^{0.4}$$

$$Re_x = \frac{U_\infty x}{V}, \quad Nu_x = \frac{h_x}{k} \Rightarrow \text{Convert to } R \text{ length scale}$$

$$\frac{h_x}{k} = 0.57 \left(\frac{U_\infty x}{V} \right)^{1/2} Pr^{0.4}$$

$$\text{For a cylinder, } U_\infty = \frac{2Vx}{R} \quad (V = \text{far field free stream velocity})$$

$$\frac{h_x}{k} = 0.57 \left(\frac{(2Vx)^2}{VR} \right)^{1/2} Pr^{0.4}$$

Multiply both sides by R

$$\frac{hR}{k} = 0.57 (2)^{1/2} \left(\frac{VR}{V} \right)^{1/2} Pr^{0.4}$$

$$Nu_R = \frac{hR}{k} = 0.81 Re_R^{1/2} Pr^{0.4}$$

$0.5 < Pr < 10, \quad 0 < Re_R < 5 \times 10^5$ (Laminar)

\Rightarrow For a cylinder near stagnation pt.

Note, we can do the same analysis for a sphere:

$$U_\infty = \frac{3Vx}{2R} \quad \text{for small } \frac{x}{R} \Rightarrow \text{Sphere (3-D)}$$

Following the same steps as above, we obtain:

$$Nu_R = 0.93 Re_R^{1/2} Pr^{0.4} \quad 0.5 < Pr < 10, \quad 0 < Re_R < 5 \times 10^5 \text{ (Laminar)}$$

\rightarrow For a sphere near stagnation point.

Note, these are not equivalent to $Nu_R = \frac{hR}{k}$!

Note both of these solutions are for $Re_R < 5 \times 10^5$ which is well beyond when vortices form ($Re_R \approx 40$). However, vortices are on the trailing edge, and these apply to the leading edge.

Blowing & Suction at the Wall (Similarity Solutions)

So far, we have always assumed that $v|_{y=0} = 0$ since our wall was impermeable.

A number of applications use suction ($v|_{y=0} < 0$) or blowing ($v|_{y=0} > 0$). For example porous surfaces or surfaces with condensation or evaporation.

Our fundamental equations don't change, mainly:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Momentum}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \text{Energy}$$

\Updownarrow Using our streamfunction formulation (η)

$$f''' + \frac{1}{2}(m+1)ff'' + m(1-f')^2 = 0 \Rightarrow \text{Momentum} \quad ①$$

$$\theta'' + \frac{1}{2}\Pr(m+1)f\theta' = 0 \Rightarrow \text{Energy} \quad ②$$

Note, our boundary conditions before were: (in momentum)

Dimensional	{	Similarity Solution ($f' = \frac{u}{U_\infty}$)
1) $u(y=0) = 0$		1) $f'(y=0) = 0 ; \eta = y \sqrt{\frac{U_\infty}{v x}}$
2) $u(y \rightarrow \infty) = U_\infty$		2) $f'(y \rightarrow \infty) = 1$
3) $v(y=0) = 0$	3) $v = \frac{1}{2} \left(\frac{v U_\infty}{x} \right)^{1/2} \cdot (\eta f' - f) \quad [\text{From mass conservat.}]$	
	at $y=0, \eta=0$ so $f(0)=0$	Pg. 48

However, now that we have arbitrary blowing or suction, our third boundary condition must change if we want to attempt a similarity solution.

Note, for an arbitrarily prescribed $v|_{y=0}$, equations ① and ② on the previous page must be solved numerically.

For our case, we seek a similarity solution knowing:

$$v = -\frac{\partial \psi}{\partial x}, \quad \psi = \sqrt{U \times U_\infty} f, \quad U_\infty = Cx^m$$

If we combine these, we obtain

$$v = -f \frac{m+1}{2} x^{(m-1)/2} \sqrt{Cv} - Cy \frac{m-1}{2} f' x^{m-1}$$

But we know for our case, at $y=0$ or $\eta=0$, $v=v_s$

$$v|_{\eta=0} = v_s = -f \frac{m+1}{2} x^{(m-1)/2} \sqrt{Cv} - \underbrace{C(0) \frac{m-1}{2} f'(0) x^{m-1}}_0$$

$$\text{We know } C = U_\infty / x^m$$

$$v_s = -\frac{m+1}{2} f(0) \frac{U_\infty}{\sqrt{U_\infty \times v}}$$

or

$$f(0) = -\frac{2}{m+1} \frac{v_s}{U_\infty} \sqrt{\frac{U_\infty x}{v}} = -\frac{2}{m+1} \frac{v_s}{U_\infty} Re_x^{1/2} \Rightarrow \text{Our new boundary condition.}$$

Since we know that f is a similarity solution, and f is a function of η only, for $\eta=0$ ($y=0$), f must be a constant for all solutions.

So to obtain a true similarity solution, our b.c.'s become:

$$f(0) = -\frac{2}{m+1} \frac{V_s}{U_\infty} Re_x^{1/2} = \text{constant} \quad ③$$

$$f'(0) = 0 \quad ④$$

$$f(\eta \rightarrow \infty) = 1 \quad ⑤$$

Equation ③ implies that to obtain a similarity solution,

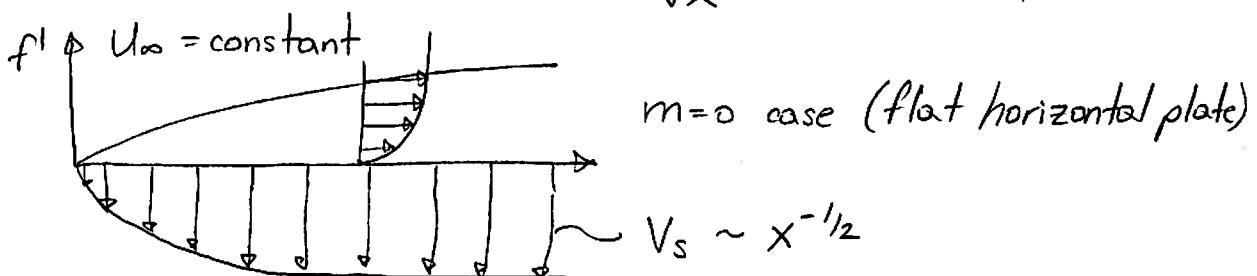
$$\begin{aligned} f(0) = \text{constant} &= -\frac{2}{m+1} \frac{V_s}{U_\infty} \sqrt{\frac{U_\infty x}{V}} \Rightarrow U_\infty = Cx^m \\ &= -\frac{2}{m+1} \frac{V_s}{Cx^m} \sqrt{\frac{Cx^{m+1}}{V}} \\ &= -\frac{2}{m+1} \frac{C^{1/2} x^{1/2} x^{m/2}}{C^{1/2} x^{m/2}} \cdot \frac{V_s}{\sqrt{V}} = -\frac{2}{m+1} C^{-1/2} x^{-(m-1)/2} \frac{V_s}{\sqrt{V}} \end{aligned}$$

$$\text{constant} = -\frac{2}{m+1} x^{-(m-1)/2} \cdot \frac{V_s}{\sqrt{V}}$$

$$V_s \sim x^{(m-1)/2} \quad \text{for this equation to work for all } x$$

The other possible similarity solution is for $m=1$ (stagnation)

For $U_\infty = \text{constant}$, $m=0$, $V_s \sim \frac{1}{\sqrt{x}}$ for similarity solution:



Typically, similarity solutions are characterized by:

$$\frac{V_s}{U_\infty} Re_x^{1/2} \equiv \text{Suction or Blowing Parameter} \quad (73)$$

We use this parameter since we know in our solution that:

$$\text{Constant} = -\frac{2}{m+1} \cdot \underbrace{\frac{V_s}{U_\infty}}_{\text{constant}} \underbrace{Re_x^{1/2}}_{\text{constant}} \Rightarrow \text{from eq. ③}$$

So for any m , and suction or blowing parameter, equation ① can be numerically solved & results tabulated.

$\frac{V_s}{U_\infty} Re_x^{1/2}$	$f''(0) = \frac{1}{2} C_{f,x} Re_x^{1/2}$	$(m=0)$
- 2.5	2.59	Suction
- 0.75	0.945	
- 0.25	0.523	
0	0.332	
+ 0.25	0.165	Impermeable wall
+ 0.375	0.094	
+ 0.5	0.036	
+ 0.619	0	Blowing or Injection
		Separation

It's worth noting the order of magnitude of V_s here:

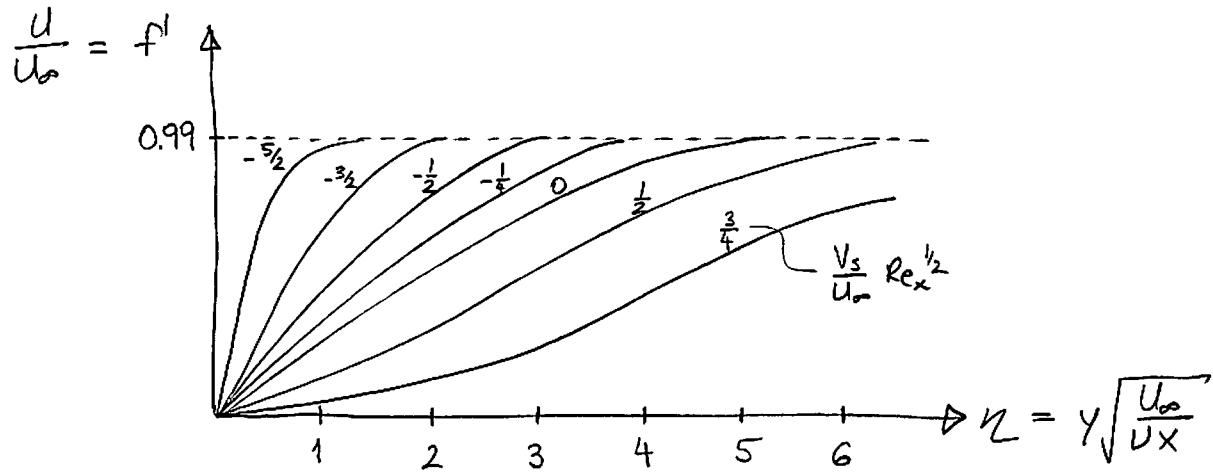
$$\frac{V_s}{U_\infty} Re_x^{1/2} \sim 1 \Rightarrow V_s \sim \frac{U_\infty}{Re_x}$$

Looking at v in our boundary layer (cons. of mass)

$$v \sim \frac{U_\infty}{Re_x^{1/2}} (2f' - f) \Rightarrow \text{We know } f' \ll f \sim 1$$

So $V_s \sim v$ \Rightarrow So for the above suction velocity results, the suction parameters are such that the suction velocity, V_s is on the same order as the transverse velocity inside the b.l., v . Note, if $\frac{V_s}{U_\infty} Re_x^{1/2} \rightarrow 0$, $V_s \ll v$ and the suction solution is not very different from the flat plate solution.

If we plot our results,



From the solutions, we see that:

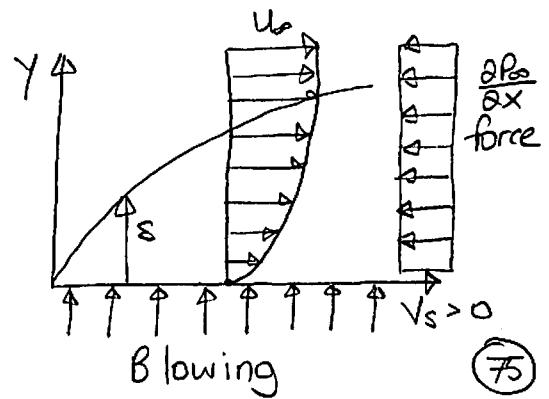
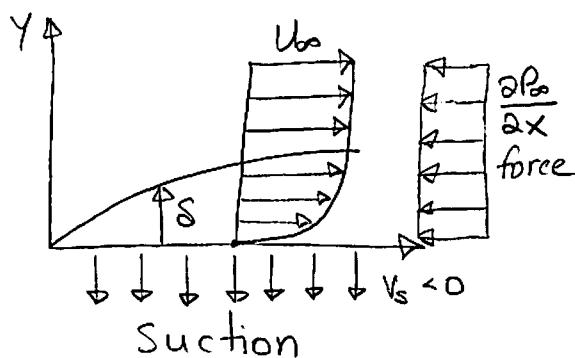
If $\frac{V_s}{U_\infty} Re_x^{1/2} < 0$ (Suction), δ decreases, C increases

If $\frac{V_s}{U_\infty} Re_x^{1/2} > 0$ (Blowing), δ increases, C decreases

At $\frac{V_s}{U_\infty} Re_x^{1/2} = 0.619 \Rightarrow$ Separation occurs $\Rightarrow \frac{\partial U}{\partial y} \Big|_{y=0} = 0$

Suction tends to increase the boundary layer attachment and delay separation. This comes at the cost of increased shear and drag. Sometimes used on aircraft wings: F-16XL
General Dynamics

Think of it as the suction pulls in the U_∞ stream closer to the wall, which can act against the $\frac{\partial P_\infty}{\partial x}$ adverse momentum close to the wall.



A simple particular solution can be obtained far from the leading edge.

Assuming $U_\infty = \text{constant}$, $m = 0$ (flat plate), $\frac{\partial P}{\partial x} = 0$
Our B.C.'s are:

$$\begin{aligned} u(y=0) &= 0 && (\text{no slip}) \\ v(y=0) &= -V_s && (\text{suction}) \\ u(y \rightarrow \infty) &= U_\infty \end{aligned}$$

One particular solution that satisfies these b.c.'s is where $u \neq f(x)$ or $u = f(y)$ only. Our momentum equation becomes:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} + v \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\text{From continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v(y) = \text{constant}$$

$$\text{Since } v|_{y=0} = V_s \Rightarrow v(y) = V_s \quad (2)$$

Substituting (2) into (1), we obtain:

$$V_s \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \text{Solvable ODE, second order, linear}$$

$$u(y) = A e^{-\lambda y} + B e^{+\lambda y} \quad \frac{\partial^2 u}{\partial y^2} + \frac{V_s}{v} \frac{\partial u}{\partial y} = 0$$

$$\lambda^2 + \frac{V_s}{v} \lambda = 0 \Rightarrow \lambda(\lambda + \frac{V_s}{v}) = 0 \Rightarrow \lambda = -\frac{V_s}{v} \text{ or } \lambda = 0$$

$$u(y) = A e^{+\frac{V_s}{v} y} + B e^{-\frac{V_s}{v} y} + C \Rightarrow \text{Since } u(y \rightarrow \infty) = U_\infty, \quad A = 0$$

$$u(y \rightarrow \infty) = U_\infty = B e^{-\frac{V_s}{v}(\infty)} + C \Rightarrow C = U_\infty$$

$$u(y=0) = 0 = B e^{0} + U_\infty \Rightarrow B = -U_\infty$$

$$\therefore \boxed{u(y) = U_\infty \left(1 - e^{-\frac{V_s y}{v}}\right)} \Rightarrow \text{Asymptotic solution far from the leading edge.}$$

Note, this solution is only valid for: $\left[\frac{V_s}{U_\infty} Re_x^{1/2} > 2 \right]$

Our previous tabulated solutions for $m=0$ will fall on this curve for large x .

Heat Transfer (Suction & Blowing)

Our energy equation remains the same as before:

$$\Theta'' + \frac{1}{2} Pr(m+1)f\Theta' = 0 ; \quad \Theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

B.C.'s : $\Theta(\eta=0) = 0$
 $\Theta(\eta \rightarrow \infty) = 1$

To solve, we substitute our hydrodynamic solution, f , into the ODE above and numerically solve it with the same b.c.'s as before. Note, f is not unique and changes for each case of blowing & suction parameter, and m .

Looking at our case of $m=0$, $V_s \sim x^{-1/2}$

$\frac{V_s}{U_\infty} Re_x^{1/2}$	$Nu/Re_x^{1/2}$		(For $m=0$)
	$Pr = 0.7$	$Pr = 0.9$	
-2.5	1.85	2.59	
-0.75	0.722	0.945	
-0.25	0.429	0.523	
0	0.292	0.332	
+0.25	0.166	0.165	
+0.375	0.107	0.0937	
+0.5	0.0517	0.0356	
+0.619	0	0	

Suction

Impermeable Wall

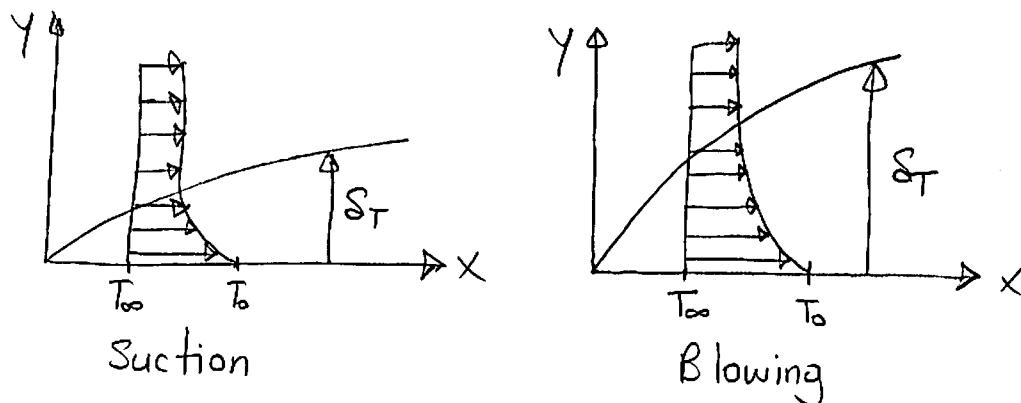
Blowing

Separation

Note the $\frac{V_s}{U_\infty} Re_x^{1/2} = 0$ solution is the Blasius-Pohlhausen solution we've done before.

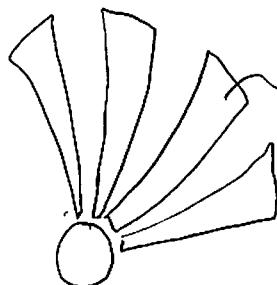
Note, $Nu_x = \frac{hx}{k} \uparrow$ for $\frac{Vs}{U_\infty} Re_x^{1/2} < 0$ (suction, S_T smaller)

$Nu_x = \frac{hx}{k} \downarrow$ for $\frac{Vs}{U_\infty} Re_x^{1/2} > 0$ (blowing, S_T larger)



Since $h \sim k \frac{\Delta T}{S_T} \Rightarrow h_{\text{suction}} > h_{\text{blowing}}$

Note, this can be a means of cooling. For example



Turbine blades on a jet or a generator

Blades are colder than the hot gases ($T_b < T_\infty$). So blowing will decrease h and cool the blade. In this case, blowing is advantageous.

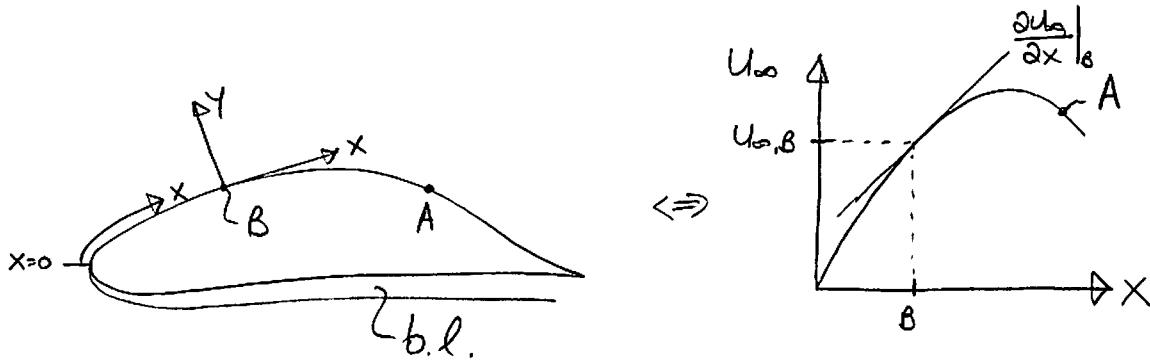
Also note, these solutions are valid for the same fluid being injected or sucked. If not \Rightarrow mass transfer is important!

Same as previously derived: pg. 68 & 69 of notes

$$\boxed{\frac{\bar{h}}{h} = \frac{2}{m+1}}$$

Local Similarity

Suppose we had an arbitrary shape with flow around it:



Assumption, conditions upstream have little influence on the b.l. behaviour at location x .

Thus, the b.l. thickness is mainly a manifestation of local conditions.

This is only an approximate analysis. From our U_∞ profile, m can be calculated for a local wedge flow:

$$\left. \begin{aligned} U_\infty &= Cx^m \\ \frac{\partial U_\infty}{\partial x} &= Cmx^{m-1} = m \frac{U_\infty}{x} \end{aligned} \right\} m = \frac{x \frac{\partial U_\infty}{\partial x}}{U_\infty}$$

We assume the b.l. thickness at x is identical to that of a wedge flow with:

$$\beta = \frac{2m}{m+1}$$

We can use this to solve for wall shear stress: $C = \mu \frac{\partial U}{\partial y} \Big|_x$

We can also solve for when separation will occur ($\beta = 0.1988$). $C \approx \mu \frac{U_\infty}{S(x)}$

Integral Methods

So far, we have only looked at similarity solutions, which are good for:

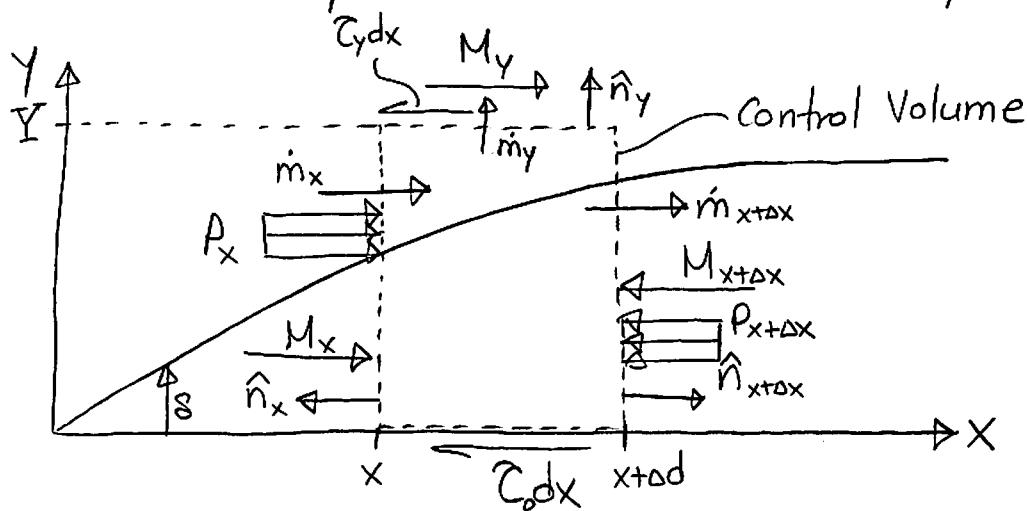
Similarity

Constant T B.C.'s
Simple flows
Simple geometries

But what if $T = f(x)$? or what about $q''|_{y=0} = \text{constant}$?

For more complex flows and geometries, it turns out integral techniques are quite useful.

Let's look at how they work. Consider a boundary layer:



Looking at each of our terms for an x-momentum balance:

$$M_x = (mu)|_x \Rightarrow m|_x = \int_0^y \rho u dy$$

$$M_x = \int_0^y \rho u^2 dy |_x \quad ①$$

$$M_{x+\Delta x} = (mu)|_{x+\Delta x} = \int_0^y \rho u^2 dy |_{x+\Delta x} \quad ②$$

$$M_y = (\rho V u)|_y \cdot \Delta x \quad ③$$

Now for our forces:

Taylor Series Expansion

$$(P_x - P_{x+\Delta x})Y = \left[P_x - \left(P_x + \frac{\partial P}{\partial x} \Big|_x \Delta x + \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \Delta x^2 + \dots \right) \right] Y \quad (4)$$

$$F_{\text{shear},0} = C dx \Big|_0 = u \frac{\partial u}{\partial y} \Big|_{y=0} \Delta x \quad (5)$$

$$F_{\text{shear},Y} = C dx \Big|_Y = u \frac{\partial u}{\partial y} \Big|_{y=Y} \Delta x \quad (6)$$

H.O.T. = higher order terms

Putting everything together, we obtain:

$$\begin{aligned} & \int_0^Y p u^2 dy \Big|_{x+\Delta x} - \int_0^Y p u^2 dy \Big|_x + (\rho v u) \Big|_Y \cdot \Delta x \\ &= - \left(\frac{\partial P}{\partial x} \Big|_x \Delta x + \text{H.O.T.} \right) Y - u \frac{\partial u}{\partial y} \Big|_0 \Delta x + u \frac{\partial u}{\partial y} \Big|_Y \Delta x \end{aligned}$$

We can Taylor expand our first two terms to obtain

$$\left[\frac{d}{dx} \int_0^Y p u^2 dy \Big|_x \Delta x + \text{H.O.T.} \right] + (\rho v u) \Big|_Y \cdot \Delta x = \sum F$$

But note we can divide both sides by Δx , and let $\Delta x \rightarrow 0$. All of our H.O.T. will drop. We obtain:

$$\frac{\partial}{\partial x} \int_0^Y p u^2 dy + (\rho v u) \Big|_Y = - \frac{\partial P}{\partial x} - u \frac{\partial u}{\partial y} \Big|_0 + u \frac{\partial u}{\partial y} \Big|_Y \quad (7)$$

The second term above is a little tricky since we don't know v . From mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \int_0^Y \frac{\partial v}{\partial y} = - \int_0^Y \frac{\partial u}{\partial x} \Rightarrow v \Big|_Y - v \Big|_0 = \int_0^Y - \frac{\partial u}{\partial x} dy$$

Now we know $v_0 = 0$ (Impermeable wall)

$$v|_y = - \int_0^Y \frac{\partial u}{\partial x} dy = - \frac{2}{\partial x} \int_0^Y u dy$$

Now our M_y term becomes:

$$\begin{aligned} M_y &= -\rho \frac{\partial}{\partial x} \left(\int_0^Y u dy \cdot u|_y \right) \Rightarrow \text{Assuming } Y > S, u|_y = U_\infty \\ &= -\rho \frac{\partial}{\partial x} \int_0^Y u dy \cdot U_\infty = -\rho \frac{\partial U_\infty}{\partial x} \int_0^Y u dy - U_\infty \rho \frac{\partial}{\partial x} \int_0^Y u dy \end{aligned}$$

Back substituting M_y into ⑦, we obtain:

$$\frac{\partial}{\partial x} \int_0^Y \rho u^2 dy - \rho \frac{\partial U_\infty}{\partial x} \int_0^Y u dy - U_\infty \rho \frac{\partial}{\partial x} \int_0^Y u dy = - \frac{\partial P}{\partial x} Y - U_\infty \frac{\partial u}{\partial y} \Big|_0 + u \frac{\partial u}{\partial y} \Big|_0$$

Again, since $Y > S$, $\frac{\partial u}{\partial y} \Big|_0 = 0$ ($u = U_\infty \neq f(y)$)

$$\boxed{\frac{\partial}{\partial x} \int_0^Y u(U_\infty - u) dy = \frac{1}{\rho} \frac{\partial P}{\partial x} Y + U_\infty \frac{\partial u}{\partial y} \Big|_0 - \frac{\partial U_\infty}{\partial x} \int_0^Y u dy} \quad ⑧$$

→ Momentum Integral Boundary Layer Eqn.

Note in ME429 we automatically assumed $U_\infty = \text{constant}$ and $\frac{\partial P_\infty}{\partial x} = 0$, so reduces to:

$$\boxed{\frac{\partial}{\partial x} \int_0^Y u(U_\infty - u) dy = U_\infty \frac{\partial u}{\partial y} \Big|_0} \quad ⑨ \Rightarrow \begin{aligned} &\text{Flat Plate} \\ &U_\infty = \text{constant}, \frac{\partial P_\infty}{\partial x} = 0 \end{aligned}$$

To solve this equation, we can first non-dimensionalize

$$\frac{u}{U_\infty} = \phi(\eta); \quad \eta = \frac{y}{S} \Rightarrow dy = S d\eta$$

Back substituting into ⑨ for a flat plate and $\frac{\partial P}{\partial x} = 0$

Note also, we will integrate from 0 to $\eta = S$ since for $\eta > S$, our integral is zero (not useful to us).

$$\frac{\partial}{\partial x} \int_0^S u(u_\infty - u) dy = U \frac{\partial u}{\partial y} \Big|_0$$

$$u = U_\infty \phi$$

$$dy = S d\eta$$

$$\frac{\partial u}{\partial y} = \frac{U_\infty \frac{\partial \phi}{\partial \eta}}{S \frac{\partial \eta}{\partial y}}$$

$$\frac{\partial}{\partial x} \int_0^1 U_\infty \phi (U_\infty - U_\infty \phi) S d\eta = U \frac{U_\infty}{S} \frac{\partial \phi}{\partial \eta} \Big|_0$$

$$\frac{\partial}{\partial x} \int_0^1 U_\infty^2 \phi (1 - \phi) S d\eta = U \frac{U_\infty}{S} \frac{\partial \phi}{\partial \eta} \Big|_0$$

Since $U_\infty = \text{constant}$ and $S \neq f(\eta)$ only $f(x)$ we take it out

$$S \frac{\partial S}{\partial x} \int_0^1 \phi (1 - \phi) d\eta = \frac{U}{U_\infty} \frac{\partial \phi}{\partial \eta} \Big|_0$$

So our equation becomes:

$$\boxed{S \frac{\partial S}{\partial x} = \frac{U}{U_\infty} \beta} ; \quad \boxed{\beta = \frac{\phi'|_0}{\int_0^1 \phi (1 - \phi) d\eta}} \quad (10)$$

We know $\beta = f(y)$ and $\beta \neq f(x)$, so we can simply integrate

$$\int_0^x S ds = \int_0^x \frac{U}{U_\infty} \beta dx$$

$$\frac{S^2}{2} \Big|_0^x = \frac{U \beta x}{U_\infty} \Rightarrow \left(\frac{S}{x} \right)^2 = 2 \beta \frac{U}{U_\infty x} \xrightarrow{\text{Rex}^{-1}} \boxed{\frac{S}{x} = \sqrt{\frac{2 \beta}{\text{Rex}}}} \quad (11)$$

We've obtained this result before with $\sqrt{2\beta} = 5.0$ (Blasius)

The way to solve momentum integral problems is to approximate the dimensionless velocity profile $\phi(\eta)$. This method is approximate but it turns out to work very well as long as the B.C.'s are matched.

Note: It was Von-Karman (Postdoc) and Prandtl (student) of Prandtl who came up with this method in 1919.

Let's assume some profiles & solve for β :

$$\phi(0) = \frac{U}{U_\infty} \Big|_{\eta=0} = 0 \quad (\text{No slip}) \quad (12)$$

$$\phi(1) = \frac{U_\infty}{U_\infty} \Big|_{\eta=1} = 1 \quad (\text{Free stream}) \quad (13)$$

$$\phi'(1) = \frac{\partial \phi}{\partial \eta} \Big|_{\eta=1} = 0 \quad (\text{no shear}) \quad (14)$$

$$\phi''(0) = \frac{\partial^2 \phi}{\partial \eta^2} \Big|_{\eta=0} = 0 \quad (\text{linear grad.}) \quad (15)$$

} Boundary Conditions

} \Rightarrow Constant wall shear stress.

Assuming the following velocity profiles:

$\phi = \eta$ \Rightarrow Satisfies (12) & (13), but not (14) or (15)

$\phi = 2\eta - \eta^2$ \Rightarrow Satisfies (12), (13) & (14), but not (15)

$\phi = \sin\left(\frac{\pi}{2}\eta\right)$ \Rightarrow Satisfies all b.c.'s

$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$ \Rightarrow Satisfies all b.c.'s

Back substituting ϕ into (10) and solving for $\sqrt{2\beta'}$

$$\beta' = \frac{\phi'|_0}{\int_0^1 \phi'(1-\phi) d\eta}$$

ϕ	$\sqrt{2\beta'}$
η	3.464
$2\eta - \eta^2$	5.477
$\sin(\pi/2\eta)$	4.795
$\frac{3}{2}\eta - \frac{1}{2}\eta^2$	4.641

All solutions are remarkably close to $\sqrt{2\beta'} = 5.0$ (Blasius exact solution!)

This shows that the integral method is particularly good at approximating the correct solution from fairly crude profile assumptions.

Let's solve for shear (C) using $\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$

$$C(x) = U \left. \frac{\partial U}{\partial y} \right|_0 \Rightarrow U = U_\infty \phi \\ \frac{dy}{dx} = 8d\eta$$

$$= U \left. \frac{U_\infty \partial \phi}{8 \partial \eta} \right|_0 \Rightarrow \left. \frac{\partial \phi}{\partial \eta} \right|_{\eta=0} = \left. \frac{3}{2} - \frac{3}{2}\eta^2 \right|_0 = \frac{3}{2}$$

$$C(x) = U \frac{U_\infty}{8} \cdot \frac{3}{2}$$

Now if we want $C_{f,x}$

$$C_{f,x} = \frac{C(x)}{\frac{1}{2} \rho U_\infty^2} = \frac{U U_\infty G}{2 \rho U_\infty^2} = \frac{3U}{8 \rho U_\infty} \left(\frac{x}{X} \right) = \frac{3U}{\left(\frac{x}{X} \right) \rho U_\infty X}$$

Multiply by $\left(\frac{x}{X} \right) \Rightarrow$ trick

$$\text{But note: } Re_x = \frac{U_\infty X}{V} \Rightarrow C_{f,x} = \frac{3}{\frac{8}{X} Re_x}$$

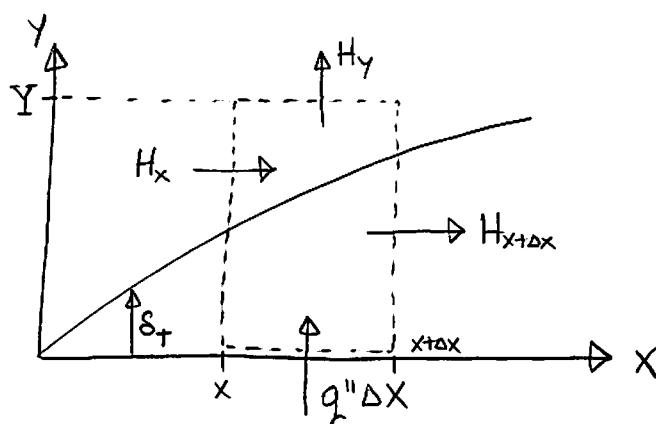
But we already showed $\left(\frac{x}{X} \right) = \left(\frac{2\beta'}{Re_x} \right)^{1/2} \Rightarrow$ Back substitute

$$C_{f,x} = \frac{3}{\sqrt{2\beta'}} \frac{1}{\sqrt{Re_x}} \Rightarrow \text{We solved above that } \sqrt{2\beta'} = 4.641$$

$C_{f,x} = \frac{C(x)}{\frac{1}{2} \rho U_\infty^2} = 0.646 Re_x^{-1/2}$	\Rightarrow Using similarity, we had: $C_{f,x} = 0.664 Re_x^{1/2}$
--	---

Energy Integral (Heat Transfer)

Similar to what we just did, we can develop an energy balance on our CV:



* Assuming viscous dissipation heating is negligible.

H_x = enthalpy flow in the x plane

Writing out our terms:

$$H_x = \left(\int_0^Y \rho c_p u T dy \right) \Big|_x \quad \text{Taylor Series Expans.}$$

$$H_{x+\Delta x} = \left(\int_0^Y \rho c_p u T dy \right) \Big|_{x+\Delta x} = \left(\int_0^Y \rho c_p u T dy \right) \Big|_x + \frac{\partial}{\partial x} \left(\int_0^Y \rho c_p u T dy \right) \Delta x + \dots$$

$$H_y = (\rho v c_p T) \Big|_y \cdot \Delta x$$

$$q'' \Delta x = -k \frac{\partial T}{\partial y} \Big|_0 \cdot \Delta x$$

Substituting all of our terms into our energy equation

$$-H_{x+\Delta x} + H_x - H_y + q'' \Delta x = 0 \quad \text{or} \quad H_{x+\Delta x} - H_x + H_y = q'' \Delta x$$

$$\left[\left(\int_0^Y \rho c_p u T dy \right) \Big|_x + \frac{\partial}{\partial x} \left(\int_0^Y \rho c_p u T dy \right) \Big|_x \Delta x + H.O.T \right] - \left(\int_0^Y \rho c_p u T dy \right) \Big|_x$$

$$+ (\rho v c_p T) \Big|_y \cdot \Delta x = -k \frac{\partial T}{\partial y} \Big|_0 \cdot \Delta x \Rightarrow \text{Divide through by } \Delta x \text{ and let } \Delta x \rightarrow 0.$$

From continuity (solved during momentum integral calculation)

$$v|_y = -\frac{\partial}{\partial x} \int_0^y u dy \Rightarrow \text{Back substitute into our } M_y \text{ term:}$$

I assumed here $\rho C_p = \text{constants}$

$$M_y = \left(\rho v C_p T \right) \Big|_y \cdot \Delta x = - \left(\rho C_p \frac{\partial}{\partial x} \int_0^y u dy \cdot T \right) \Big|_y \cdot \Delta x$$

$$M_y = - \left[\rho C_p \frac{\partial T}{\partial x} \Big|_y \int_0^y u dy + \rho C_p T \Big|_y \int_0^y \frac{\partial u}{\partial x} dy \right] \cdot \Delta x$$

Back substituting (dropping the Δx since we already canceled it).

$$\begin{aligned} \frac{\partial}{\partial x} \left(\int_0^y \rho C_p u T dy \right) \Big|_x &= \rho C_p T \Big|_y \frac{\partial}{\partial x} \int_0^y u dy - \rho C_p \frac{\partial T}{\partial x} \Big|_y \int_0^y u dy \\ &= -k \frac{\partial T}{\partial y} \Big|_y. \end{aligned}$$

Assuming $\rho C_p = \text{constant}$, & knowing $\alpha = \frac{k}{\rho C_p}$

$$\boxed{\frac{\partial}{\partial x} \int_0^y u (T_\infty - T) dy = \alpha \frac{\partial T}{\partial y} \Big|_y - \frac{\partial T}{\partial x} \int_0^y u dy}$$

↳ Integral Boundary Layer Equation for Energy

Usually, we assume two things to simplify:

- 1) Integrate to $y = S_T$ since for $y > S_T \Rightarrow$ integrals become 0.
- 2) The $T_\infty = \text{constant}$. For the above equation, it doesn't need to be however.

$$\boxed{\frac{\partial}{\partial x} \int_0^y u (T_\infty - T) dy = \alpha \frac{\partial T}{\partial y} \Big|_y} \Rightarrow T_\infty = \text{constant}$$

①

So let's try the same approach we used for the momentum integral:

$$\Theta(\eta_T) = \frac{T - T_0}{T_\infty - T_0} ; \quad \eta_T = \frac{y}{S_T} \Rightarrow d\theta = (T_\infty - T_0) dT \\ dy = S_T d\eta_T$$

We see right away that:

$$\frac{T - T_\infty}{T_0 - T_\infty} = 1 - \frac{T - T_0}{T_\infty - T_0} = 1 - \Theta$$

Back substituting into ① \Rightarrow Also $\phi = \frac{U}{U_\infty}$

$$\frac{2}{2x} \int_0^1 U_\infty \phi (1-\Theta) (T_0 - T_\infty) S_T d\eta_T = \left. \frac{\alpha}{S_T} \frac{\partial \Theta}{\partial \eta_T} \right|_0 \cdot (T_0 - T_\infty)$$

$$\frac{2}{2x} \int_0^1 S_T \phi (1-\Theta) \partial \eta_T = \left. \frac{\alpha}{U_\infty S_T} \frac{\partial \Theta}{\partial \eta_T} \right|_0 \quad ②$$

Our B.C.'s are:

$$\Theta(\eta_T=0) = 0 \quad (\text{Wall temperature, } T_0)$$

$$\Theta(\eta_T=1) = 1 \quad (\text{Free stream temperature, } T_\infty)$$

$$\Theta'(\eta_T=1) = 0 \quad (\text{No temp. gradient at b.l. edge})$$

$$\Theta''(\eta_T=0) = 0 \quad (\text{Linear Temp. profile at wall})$$

Using the identical clever math trick as before:

$$\Theta(\eta^*) = \frac{S}{S_T} \phi ; \quad \eta^* = \eta \Pr^{1/3}$$

We can check the rationality of our assumption

$$\left[\phi = \frac{3}{2} \eta - \frac{1}{2} \eta^3 = \frac{3}{2} \cdot \frac{y}{S} - \frac{1}{2} \left(\frac{y}{S} \right)^3 \right] \cdot \frac{S}{S_T}$$

$$\Theta = \frac{3}{2} \left(\frac{y}{S_T} \right) - \frac{1}{2} \left(\frac{y^3}{S^2 S_T} \right) \Rightarrow \text{Note here we can simplify}$$

We've implicitly assumed that $U \propto U_\infty$, i.e. $Pr \approx 1$ or $Pr \gg 1$. Because of this assumption, we can safely say that $\delta > \delta_T$. So comparing our two terms near the wall (where it's important)

$$\frac{\frac{3}{2} \left(\frac{y}{\delta_T} \right)}{\frac{1}{2} \left(\frac{y^3}{\delta^2 \delta_T} \right)} = \frac{3\delta^2}{y^2} \gg 1 \Rightarrow \text{Since for energy solution } (\theta) \\ 0 < y < \delta_T, y_{\max} = \delta_T \\ = \frac{3\delta^2}{\delta_T^2} \gg 1 \text{ since } Pr \approx 1 \text{ or } Pr \gg 1$$

So this allows us to safely say that the second term in our θ solution is negligible compared to the first.

Note, this is also possible because our B.C.'s in non-dimensional momentum (ϕ) and energy (θ) are identical.

Now we have the following:

$$\theta(n^*) = \frac{3}{2} \frac{y}{\delta_T} \Rightarrow \text{Back substitute into } ② \quad \left(\phi = \frac{\delta_T}{\delta} \theta \right)$$

$$\frac{2}{\partial x} \left(\frac{\delta_T^2}{\delta} \right) \int_0^1 \theta(1-\theta) d\eta_T = \frac{\alpha}{U \delta_T} \left(\frac{U}{U_\infty} \theta' \Big|_0 \right)$$

$$\frac{2}{\partial x} \left(\frac{\delta_T^2}{\delta} \right) = \frac{1}{Pr \delta_T} \cdot \frac{U}{U_\infty} \underbrace{\left[\frac{\theta' \Big|_0}{\int_0^1 \theta(1-\theta) d\eta_T} \right]}_{\beta}$$

Rearranging:

$\beta \Rightarrow$ We've defined this before!

$$\delta_T \frac{2}{\partial x} \left(\frac{\delta_T^2}{\delta} \right) = \underbrace{\frac{1}{Pr} \left(\frac{U}{U_\infty} \beta \right)}_{= \delta \frac{\partial \delta}{\partial x}} \quad (\text{Derived this before})$$

$$S_T \frac{\partial}{\partial x} \left(\frac{S_T^2}{S} \right) = \frac{S}{Pr} \frac{\partial S}{\partial x} \cdot \left(\frac{S''^2}{S^{3/2}} \right) \Rightarrow \text{Integrate both sides}$$

$$\int_{x_0}^x \frac{S_T}{S^{3/2}} \frac{\partial}{\partial x} \left(\frac{S_T^2}{S} \right) = \int_{x_0}^x \frac{1}{Pr} S^{1/2} \frac{\partial S}{\partial x} \Rightarrow \text{Note, my bounds of integration are } x_0 \text{ to } x. \\ I \text{ will explain later!}$$

$$\Rightarrow \underbrace{\int_{x_0}^x \frac{S_T}{S^{3/2}} d(S_T^2)}_{= \int_{x_0}^x \frac{1}{Pr} S^{1/2} dS}$$

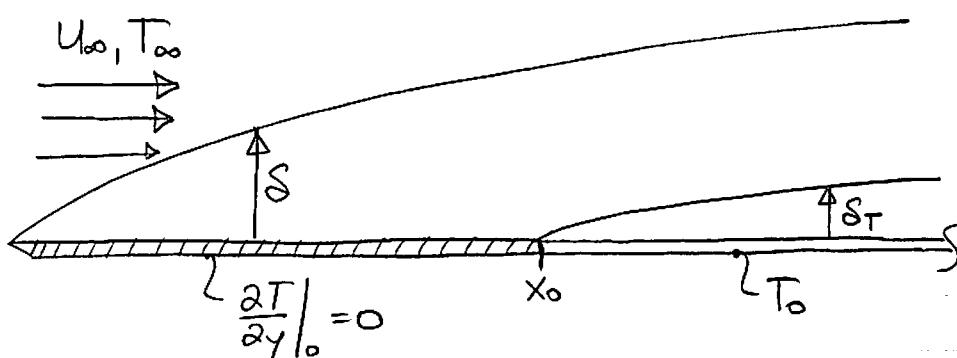
$$\int_{x_0}^x \frac{S_T}{S^{3/2}} 2S_T dS_T = \int_{x_0}^x \frac{2S_T^2}{S^{3/2}} dS_T = \frac{2}{3} \frac{S_T^3}{S^{3/2}} \Big|_{x_0}^x$$

Our second integral is more trivial : $\int_{x_0}^x S^{1/2} dS = \frac{2}{3} S^{3/2} \Big|_{x_0}^x$

$$\frac{2}{3} \frac{S_T^3}{S^{3/2}} \Big|_{x_0}^x = \frac{2}{3} S^{3/2} \Big|_{x_0}^x$$

$$\left(\frac{S_T^2}{S} \right)^{3/2} = \frac{1}{Pr} S^{3/2} \left[1 - \left(\frac{S_0}{S} \right)^{3/2} \right]$$

So the reason we integrated from x_0 instead of 0 is because it allows us to get a much more general and powerful solution. The physical picture is the following:



So for this situation, at x_0 , $S_T = 0$ and $S_0 = C\sqrt{x}$.

Note: $\frac{S}{x} \sim \frac{C}{Re_x^{1/2}} \Rightarrow S \sim \sqrt{x}$

$$\left(\frac{\delta_T}{\delta}\right)^3 = \frac{1}{Pr} \left(1 - \left(\frac{x_0}{x}\right)^{3/4}\right) \Rightarrow I \text{ subbed in } S_0 = C\sqrt{x_0} \text{ & } \delta = C\sqrt{x}$$

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3} \cdot \left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3}}$$

\Rightarrow Note, we didn't assume a Θ profile and got this! Only assumed $\delta \sim \sqrt{x}$ which is obvious!

Note, if we check our result with $x_0 = 0$, we obtain the same familiar result as before:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}} \Rightarrow x_0 = 0, Pr \geq 1.$$

So how do we calculate heat transfer?

$$q'' \Big|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_0 = -\frac{k}{\delta_T} \cdot \frac{\partial \Theta}{\partial n_T} \cdot \underbrace{(T_\infty - T_0)}_{-\Delta T} \Rightarrow \text{Remember from before:}$$

$$\partial \Theta = (T_\infty - T_0) \partial T$$

$$2y = \delta_T \partial n_T$$

For $x > x_0$

$$q'' \Big|_{y=0} = \frac{k \Delta T}{\delta_T} \Theta' \Big|_0 = \frac{k \Delta T}{x} \frac{\Theta' \Big|_0}{\left(\frac{\delta_T}{\delta}\right) \left(\frac{\delta}{x}\right)}$$

We know that:

$$\frac{\delta_T}{\delta} = Pr^{-1/3} \left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \quad \text{and} \quad \frac{\delta}{x} = \frac{\sqrt{2\beta}}{Re_x^{1/2}} \Rightarrow \text{Back substitute:}$$

$$q'' \Big|_{y=0} = \left(\frac{k \Delta T}{x}\right) \frac{Pr^{1/3} Re_x^{1/2} \Theta' \Big|_0}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \sqrt{2\beta}} \Rightarrow \text{Expand this by substituting } \beta$$

$$\boxed{q'' \Big|_{y=0} = \left(\frac{k \Delta T}{x}\right) \left[\frac{\Theta' \Big|_0}{2} \int_0^1 \Theta(1-\Theta) d n_T \right]^{1/2} \cdot \frac{Re_x^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}}$$

To solve, we need to assume a temperature profile $\Theta(n_T)$.

However there is something nice we can use about this solution and our previous similarity solutions:

$$\left[\frac{\Theta'|_0}{2} \int_0^1 \Theta(1-\Theta) d\eta_T \right]^{1/2} \neq f(x_0)$$

We know that in our limit of $x_0 = 0$, this term must be 0.332 (similarity solution). Hence:

$$\boxed{\left[\frac{\Theta'|_0}{2} \int_0^1 \Theta(1-\Theta) d\eta_T \right]^{1/2} = 0.332} \Rightarrow \text{Try it for a few profiles.}$$

Let's see how good we are. We already assumed before:

$$\Theta = \frac{3}{2} \eta_T - \frac{1}{2} \eta_T^3$$

$$\Theta'|_0 = \frac{3}{2}$$

$$\begin{aligned} \int_0^1 \Theta(1-\Theta) d\eta_T &= \int_0^1 \left(\frac{3}{2} \eta_T - \frac{1}{2} \eta_T^3 \right) \left(1 - \frac{3}{2} \eta_T + \frac{1}{2} \eta_T^3 \right) d\eta_T \\ &= \int_0^1 \left(\frac{3}{2} \eta_T - \frac{9}{4} \eta_T^2 + \frac{1}{3} \eta_T^3 + \frac{3}{2} \eta_T^4 - \frac{1}{4} \eta_T^6 \right) d\eta_T \\ &= \left[\frac{3}{4} - \frac{9}{12} + \frac{3}{10} - \frac{1}{8} - \frac{1}{28} \right] = 0.13928 \end{aligned}$$

$$\left[\frac{\Theta'|_0}{2} \int_0^1 \Theta(1-\Theta) d\eta_T \right]^{1/2} = \left[\frac{3}{4} \left(\frac{3}{10} - \frac{1}{8} - \frac{1}{28} \right) \right]^{1/2} = 0.3232$$

So with $\Theta = \frac{3}{2} \eta_T - \frac{1}{2} \eta_T^3 \Rightarrow$ Our constant is 0.3232!

So finally, our solution becomes:

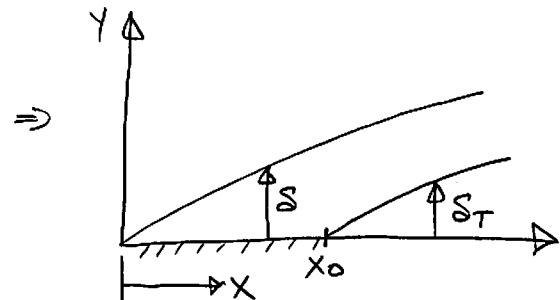
$$q''|_{y=0} = 0.332 \frac{k\Delta T}{x} \cdot \frac{Re^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$

$$h = \frac{q''|_{y=0}}{\Delta T} = \frac{0.332k}{x} \dots$$

$$Nu_x = 0.332 \frac{Re^{1/2} Pr^{1/2}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}} \Rightarrow \text{Laminar}$$

\Rightarrow From $Nu_x = \frac{hx}{k} = \frac{q''|_0}{\Delta T} \cdot \frac{x}{k}$

$\Rightarrow U_\infty, \Delta T = \text{const.}$

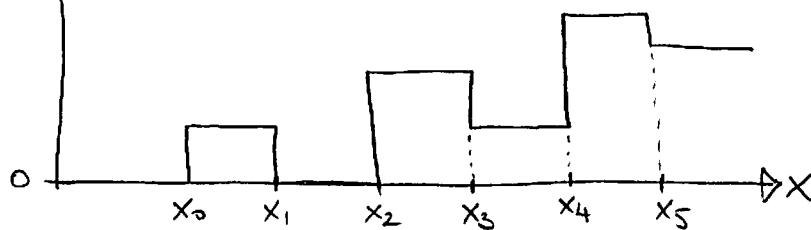


Our result kind of makes sense, however be careful. It is valid for $x > x_0$. For $x < x_0$, we've defined the plate to be adiabatic, so our solution gives us non-physical answers.

Arbitrary Varying Temperature Difference

What if instead of just one simple temperature jump like above, we had an arbitrarily varying wall temperature with multiple jumps?

$$(T_0 - T_{x_0}) \uparrow$$



Looking at our energy equation again reveals the solution:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \text{Linear Differential Equation}$$

We can use Duhamel's Theorem to say that superposition of any number of stepwise variations is a valid solution.

What Duhamel's Theorem implies is the following:

$$T = \sum_i^n T_i \Rightarrow T_i = \text{a particular solution to the energy equation for specific boundary cond.}$$

Try substituting this in, you will see that it works out.

At the surface, we know:

$$q''|_o = -k \frac{\partial T}{\partial y}|_o = -k \sum_i^n \frac{\partial T_i}{\partial y}|_o$$

Here, we can invoke that each particular solution corresponds to a temperature jump ΔT_i . For each particular solution, we can say:

$$-k \frac{\partial T_i}{\partial y}|_o = h_i \Delta T_i$$

Hence, for the full solution, we can say:

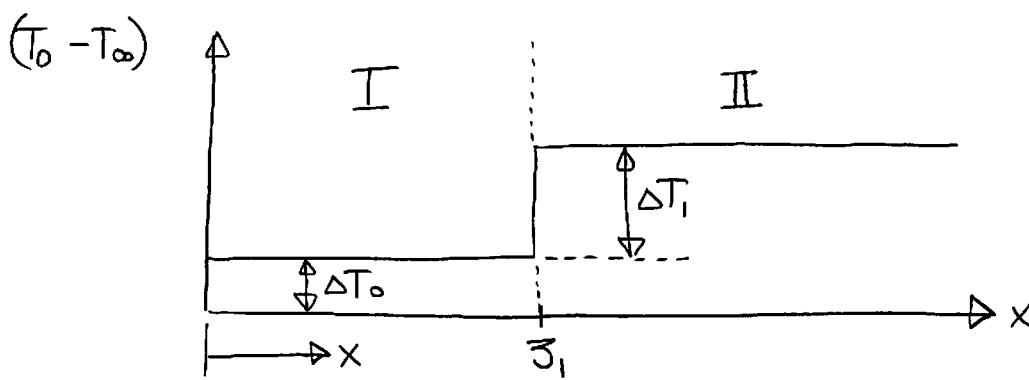
$$q'' = \sum_i^n h_i \Delta T_i$$

Translating this to our previous solution, we say:

$$q'' = \sum_i^n h(x, x_{o,i}) \Delta T_i \quad ①$$

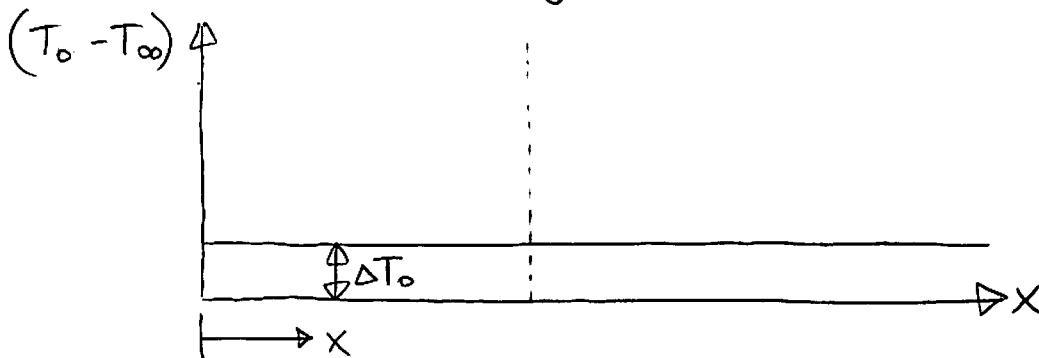
h = heat transfer coefficient which describes heat flow at location x when only one step ΔT_i in the wall temperature occurs at location $x_{o,i}$.

Note, this method of solution (superposition) can also be applied to other linear ODEs & PDEs, (i.e., wave equation).

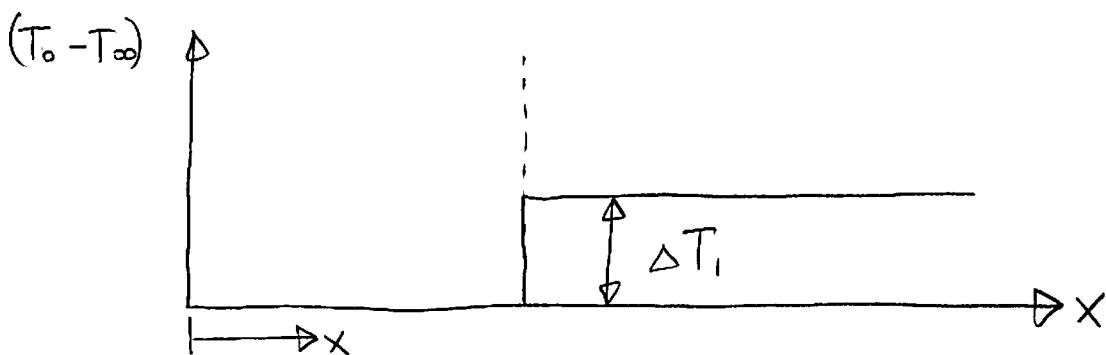


* $\bar{z} = x_0$
 ↑
 change of
 notation from
 here.

\Updownarrow Equivalent to



+



In general, we can say:

$$(T - T_{\infty}) = \sum_{j=0}^n \Delta T_j \cdot f(x, y, \bar{z}_j) \quad ; \quad \bar{z}_n < x < \bar{z}_{n+1}$$

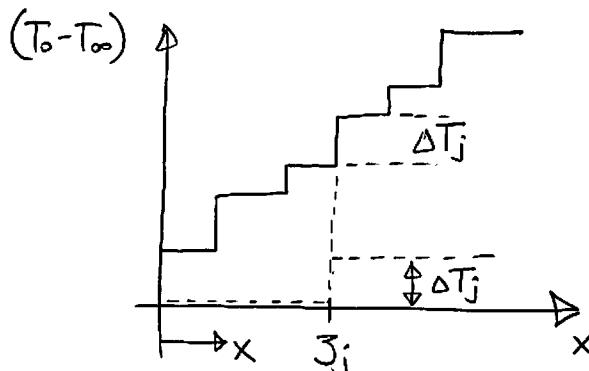
$$(T - T_{\infty}) = (T_0 - T_{\infty}) f(x, y, \bar{z})$$

Similarly: (from equation ① on the previous page)

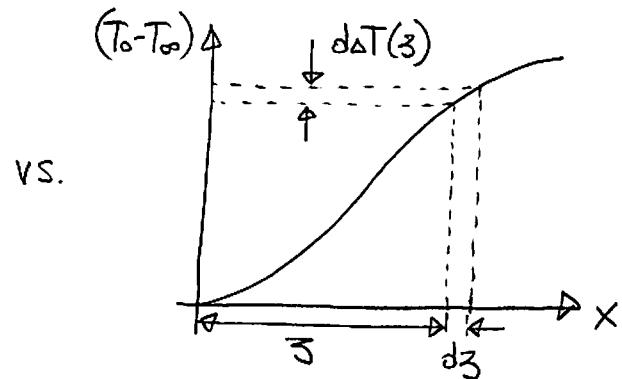
$$q''|_0 = 0.332 \left(\frac{k}{x} \right) Re_x^{1/2} Pr^{1/3} \sum_{j=0}^n \frac{\Delta T_j}{\left[1 - \left(\frac{\bar{z}_j}{x} \right)^{3/4} \right]^{1/3}} \quad ; \quad \bar{z}_n < x < \bar{z}_{n+1}$$

↳ $U_{\infty} = \text{constant}$; Laminar; $\Delta T_j = \text{constant}$

This is valid for a stepwise variation in wall temperature. What if the variation is continuous?



Stepwise (Eq. ②)



Continuous

For the continuous case, we can change ① to:

$$q''|_o = \int_0^x h(x, z) d\Delta T(z)$$

To represent this integral in terms of our independent variable, z , we use:

$$d\Delta T(z) = \frac{d\Delta T(z)}{dz} \cdot dz$$

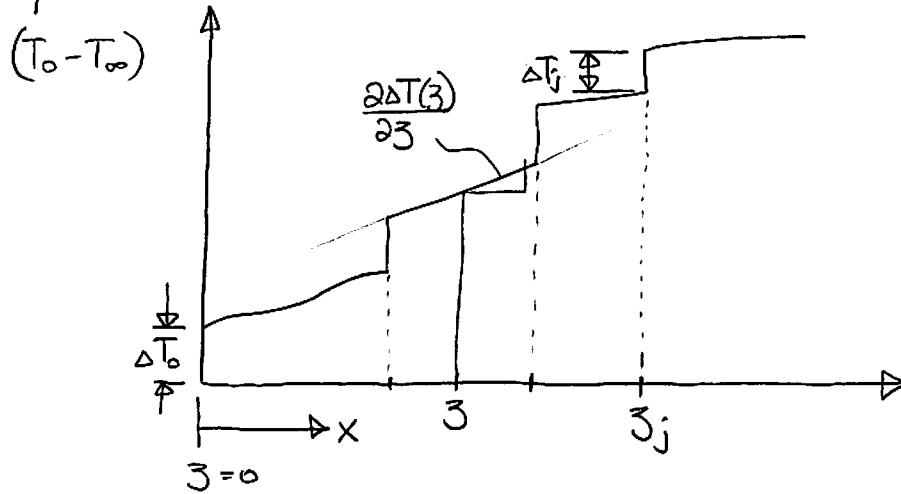
$$q''|_o = \int_0^x h(x, z) \frac{d\Delta T(z)}{dz} dz$$

Or in our notation: $\frac{d\Delta T(z)}{dz} = \frac{d(T_w - T_\infty)}{dz}$

$$q''|_o = 0.332 \left(\frac{k}{x} \right) Re_x^{1/2} Pr^{1/3} \int_0^x \frac{\frac{d(T_w - T_\infty)}{dz}}{\left[1 - \left(\frac{z}{x} \right)^{3/4} \right]^{1/3}} dz$$

↳ Laminar; $U_\infty = \text{constant}$; Continuous temperature difference variation.

For a combination of temperature jumps and continuous temp. increase:



We use the following to solve:

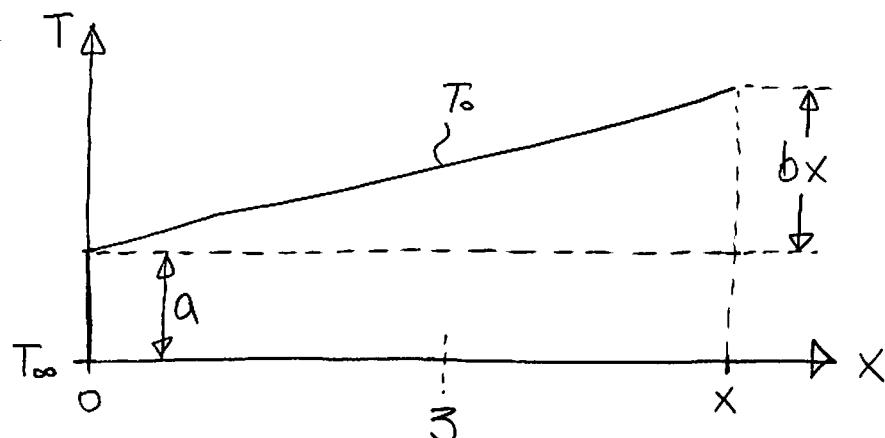
$$q''|_0 = \int_0^x h(x, z) \frac{\partial \Delta T(z)}{\partial z} dz + \sum_0^n h(x, z_j) \Delta T_j$$

$\hookrightarrow U_\infty = \text{constant}$; Laminar; Continuous & Discrete ΔT variation

Reminder:

$$h(x, z) = \frac{0.332 k}{x} Pr^{1/3} Re_x^{1/2} \left[1 - \left(\frac{z}{x} \right)^{3/4} \right]^{-1/3}$$

Example | Plate with a step and linear surface-temp. variation:



$$T_0 = T_\infty + a + bx$$

$$(T_0 - T_\infty) = a + bx$$

$$\frac{\partial \Delta T}{\partial x} = b = \frac{\partial \Delta T(3)}{\partial z}$$

Since we have a step increase (at leading edge) and a linear increase, we must use a hybrid approach:

$$q''|_o = \int_0^x h(x, z) \frac{\partial \Delta T(3)}{\partial z} dz + \sum_0^n h(x, z_j) \Delta T_j$$

For our case:

$$h(x, z) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2} \left[1 - \left(\frac{z}{x} \right)^{3/4} \right]^{-1/3}; \quad \frac{\partial \Delta T(3)}{\partial z} = b$$

$$h(x, z_j) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2} \left[1 - \left(\frac{z_0}{x} \right)^{3/4} \right]^{-1/3} \Rightarrow z_0 = 0$$

so our solution becomes:

$$\Delta T_j = a$$

$$h(x, z_0) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}$$

Putting everything together, we obtain:

$$q''|_o = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2} \left\{ \int_0^x \left[1 - \left(\frac{z}{x} \right)^{3/4} \right]^{-1/3} \cdot b dz + a \right\}$$

To solve these problems, we can use a powerful trick:

$$\boxed{\int_0^1 z^{m-1} (1-z)^{n-1} dz = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}} \Rightarrow \begin{aligned} \Gamma &= \text{gamma fund.} \\ &\Rightarrow \text{Your new best friend} \end{aligned}$$

For our case, we first need to get our integral in this form:
We will use a change of variables:

$$s = \frac{z}{x} \Rightarrow \int_0^x \left[1 - \left(\frac{z}{x} \right)^{3/4} \right]^{-1/3} dz = \int_0^1 \left[1 - s^{3/4} \right]^{-1/3} ds$$

$$ds \cdot x = dz \Rightarrow$$

Now we can do one more change of variables:

$$\begin{aligned}\lambda &= 1 - s^{\frac{3}{4}} \Rightarrow (1-s^{\frac{3}{4}})^{\frac{1}{3}} = \lambda^{\frac{1}{3}} \\ s &= (1-\lambda)^{\frac{4}{3}} \Rightarrow ds = -\frac{4}{3} (1-\lambda)^{\frac{1}{3}} d\lambda\end{aligned}\quad \left. \begin{array}{l} \text{Substitute back in} \end{array} \right\}$$

$$\int_0^1 \frac{x ds}{(1-s^{\frac{3}{4}})^{\frac{1}{3}}} = \int_1^0 \frac{x \left(-\frac{4}{3} (1-\lambda)^{\frac{1}{3}}\right) d\lambda}{\lambda^{\frac{1}{3}}} = A$$

$$\text{But since } \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$A = \int_0^1 \left(\frac{4}{3}x\right) \frac{(1-\lambda)^{\frac{1}{3}}}{\lambda^{\frac{1}{3}}} d\lambda \Rightarrow \text{Looks a lot like what we need}$$

$$= \left(\frac{4}{3}x\right) \int_0^1 \lambda^{\frac{1}{3}} (1-\lambda)^{\frac{1}{3}} d\lambda = \left(\frac{4}{3}x\right) \int_0^1 \lambda^{\frac{2}{3}-1} (1-\lambda)^{\frac{4}{3}-1} d\lambda$$

So $m = \frac{2}{3}$, $n = \frac{4}{3} \Rightarrow$ Using a Gamma function table, or Wolfram alpha (online):

$$\Gamma\left(\frac{2}{3}\right) = 1.4, \quad \Gamma\left(\frac{4}{3}\right) = 0.89, \quad \Gamma\left(\frac{2}{3} + \frac{4}{3}\right) = 1.0$$

$$\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)} = 1.246$$

So our integral term becomes:

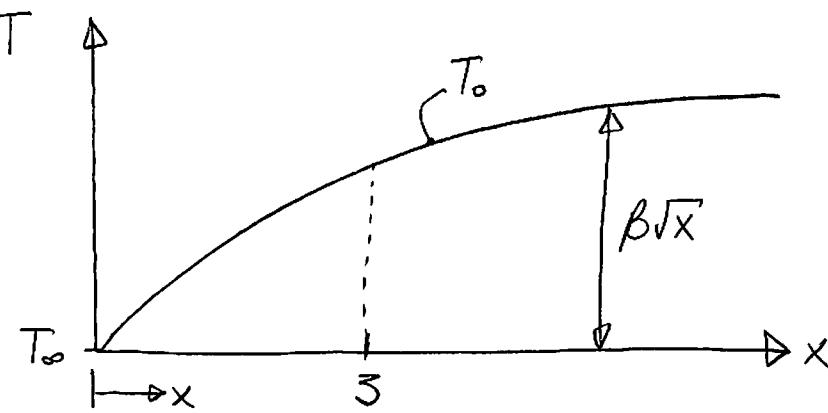
$$\int_0^x \left[1 - \left(\frac{3}{x}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}} dz = \left(\frac{4}{3}x\right)(1.246) = 1.661x$$

Back substituting:

$$q''|_0 = 0.332 \left(\frac{b}{x}\right) \Pr^{\frac{1}{3}} \operatorname{Re}^{\frac{1}{2}} (1.661bx + a)$$

\Rightarrow Result makes sense.
If $b=0$, we obtain our old solution.

Example #2 | Continuous temp. change at the wall from leading edge with $(T_0 - T_\infty) = \beta \sqrt{x}$



$$T_0 = T_\infty + \beta \sqrt{x}$$

$$\frac{dT(3)}{dx} = \frac{1}{2} \beta \frac{1}{\sqrt{x}} \Rightarrow \frac{\partial T(3)}{\partial 3} = \frac{\beta}{2\sqrt{3}} \quad (\text{since } x=3 \text{ in our domain})$$

$$h(x, 3) = \frac{0.332 k}{x} \Pr^{1/3} \operatorname{Re}_x^{1/2} \left[1 - \left(\frac{3}{x} \right)^{3/4} \right]^{-1/3}$$

$$q''|_0 = \int_0^x h(x, 3) \frac{\partial T(3)}{\partial 3} d3 \Rightarrow \text{Back substitute the above expressions}$$

$$q''|_0 = \frac{0.332 k}{2x} \Pr^{1/3} \operatorname{Re}_x^{1/2} \int_0^x \frac{\beta d3}{3^{1/2} \left[1 - \left(\frac{3}{x} \right)^{3/4} \right]^{1/3}}$$

We can use the exact same change of variable here:

$s = \frac{3}{x} \Rightarrow$ since $x=3$ in our domain, our integral bounds go from $[0, x]$ to $[0, 1]$

$$A = \int_0^x \frac{\beta d3}{3^{1/2} \left[1 - \left(\frac{3}{x} \right)^{3/4} \right]^{1/3}} = \int_0^1 \frac{\beta(x ds)}{3^{1/2} \left(1 - s^{3/4} \right)^{1/3}} \Rightarrow \text{Note } \frac{x}{3^{1/2}} = \frac{\left(\frac{x}{3} \right)^{1/2} \cdot x^{1/2}}{s^{1/2}}$$

Doing another change of variable now: (just like before)

$$\begin{aligned}\lambda &= \left(1 - s^{\frac{3}{4}}\right) \\ s &= (1-\lambda)^{\frac{4}{3}} \Rightarrow ds = -\frac{4}{3}(1-\lambda)^{\frac{1}{3}} d\lambda \\ s^{\frac{1}{2}} &= (1-\lambda)^{\frac{2}{3}}\end{aligned}$$

} Back substitute into our integral

$$\begin{aligned}A &= \int_0^1 \frac{\beta x^{\frac{1}{2}} ds}{s^{\frac{1}{2}} (1-s^{\frac{3}{4}})^{\frac{1}{3}}} = \int_1^0 \frac{(\beta x^{\frac{1}{2}})(-\frac{4}{3}(1-\lambda)^{\frac{1}{3}})}{(1-\lambda)^{\frac{2}{3}} \lambda^{\frac{1}{3}}} d\lambda \\ &= \frac{4}{3} \beta x^{\frac{1}{2}} \int_0^1 \frac{(1-\lambda)^{\frac{1}{3}}}{(1-\lambda)^{\frac{2}{3}} \lambda^{\frac{1}{3}}} d\lambda = \frac{4}{3} \beta x^{\frac{1}{2}} \int_0^1 (1-\lambda)^{-\frac{1}{3}} \lambda^{-\frac{1}{3}} d\lambda\end{aligned}$$

This now looks a lot like our Gamma integral!

$$A = \frac{4}{3} \beta x^{\frac{1}{2}} \int_0^1 \lambda^{\left(\frac{2}{3}-1\right)} (1-\lambda)^{\left(\frac{2}{3}-1\right)} d\lambda \Rightarrow m = \frac{2}{3}, n = \frac{2}{3}$$

$$\Gamma\left(\frac{2}{3}\right) = 1.355, \quad \Gamma\left(\frac{4}{3}\right) = 0.89$$

$$A = \frac{4}{3} \beta x^{\frac{1}{2}} \cdot \left(\frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} \right) = \frac{4}{3} \beta x^{\frac{1}{2}} (2.06)$$

$$q''|_0 = 0.332 \left(\frac{k}{x}\right) Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}} \beta x^{\frac{1}{2}} \left(\frac{1}{2} \cdot \frac{4}{3} \cdot 2.06\right)$$

But note that $\beta x^{\frac{1}{2}} = \Delta T(3)$ or $\Delta T(x)$, so

$$q''|_0 = 0.455 \left(\frac{k}{x}\right) Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}} \cdot \Delta T$$

$$h_x = \frac{q''|_0}{\Delta T(x)} = 0.455 \left(\frac{k}{x}\right) Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}}$$

$$Nu_x = \frac{h_x x}{k} = 0.455 Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}}$$

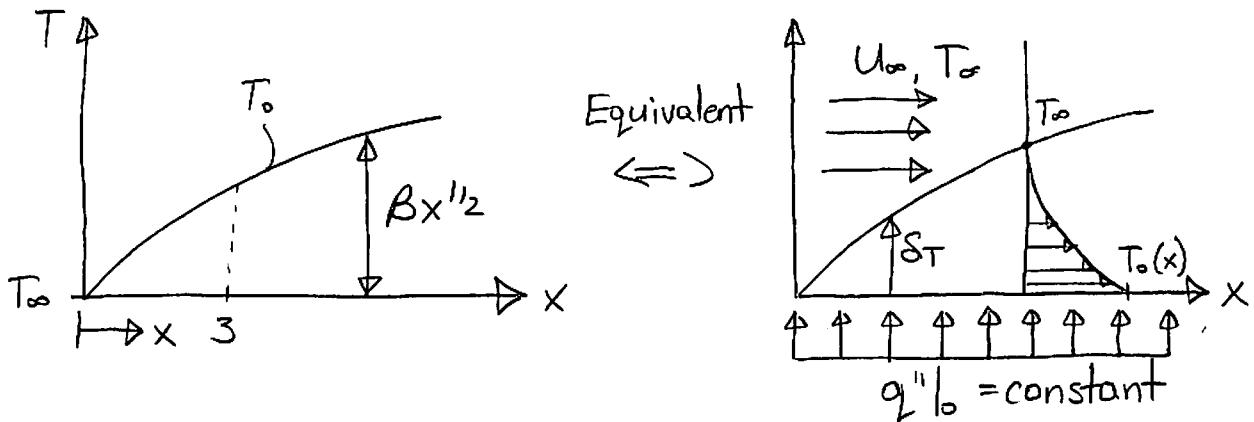
} Laminar, $U_\infty = \text{constant}$, $q''|_0 = \text{const.}$

Note however for our case, $\Delta T = \beta \sqrt{x} \Rightarrow$ Let's sub it in and simplify.

$$\begin{aligned} q''|_0 &= 0.455 \left(\frac{k}{x} \right) \Pr^{1/3} \text{Re}_x^{1/2} \Delta T \\ &= 0.455 \left(\frac{k}{x} \right) \Pr^{1/3} \left(\frac{U_\infty x}{V} \right)^{1/2} \cdot \beta x^{1/2} \end{aligned}$$

$$\boxed{q''|_0 = 0.455 k \Pr^{1/3} \beta \left(\frac{U_\infty}{V} \right)^{1/2}} \neq f(x) !$$

So we have just solved the constant heat flux boundary condition case by accident! This implies:



So now you know how the wall temperature varies for a prescribed uniform heat flux b.c. \Rightarrow it's $T_0 \sim \sqrt{x}$

So what would \bar{h} be?

$$\bar{h} = \frac{q''|_0}{\Delta T} \Rightarrow \text{Look on page 60 of notes} \Rightarrow \bar{h} = \frac{q''|_0}{\frac{1}{L} \int_0^L \Delta T dx}$$

$$\frac{1}{L} \int_0^L B\sqrt{x} dx = \frac{1}{L} \frac{2}{3} B \left[x^{3/2} \right]_0^L = \frac{1}{L} \frac{2}{3} B L^{3/2} = \frac{2}{3} B L^{1/2}$$

$$\bar{h} = \frac{q''|_0}{\frac{2}{3} B L^{1/2}} = \frac{0.455 k \Pr^{1/3} \beta \left(\frac{U_\infty}{V} \right)^{1/2}}{\frac{2}{3} B L^{1/2}} = 0.6825 \left(\frac{k}{L} \right) \Pr^{1/3} \text{Re}_L^{1/2}$$

$$\boxed{\overline{\text{Nu}}_L = \frac{\bar{h} L}{k} = 0.6825 \Pr^{1/3} \text{Re}_L^{1/2}}$$

\hookrightarrow Laminar, $U_\infty = \text{constant}$, $q''|_0 = \text{constant}$

Viscous Energy Dissipation

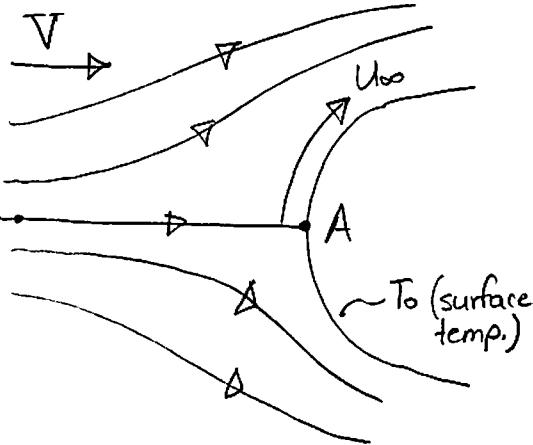
So far, we have assumed that $E_c \ll 1$ ($= \frac{U_\infty^2}{\rho(T_0 - T_\infty)}$) \Rightarrow pg 61 or that heating of the fluid through viscous dissipation is negligible compared to heating from the wall.

For convection at high velocities, we need to consider 2 things:

- 1) Conversion of mechanical energy into thermal energy, resulting in temperature variation in the fluid,
- 2) Variation of fluid properties due to temperature variation.

Let's look at consideration #1 first.

For a reversible process such as stagnation points in potential flow, we already know how to handle it:



From before:

$$\frac{1}{2} \rho V^2 + P = \text{constant}$$

$$P_A = P_\infty + \frac{1}{2} \rho V^2 \quad (\text{via Bernoulli along a streamline})$$

But for a gas with isentropic behavior $PV = nRT$, so we can say:

$$\frac{1}{2} \rho V^2 + \rho C_p T = \text{constant} \quad (\text{same streamline})$$

$$T_A = T_\infty + \frac{V^2}{2C} \Rightarrow T_A \text{ is the gas temp. at A.}$$

So to solve high velocity stagnation point problems (reversible) it's easy:

$$\left. \frac{\partial T}{\partial x} \right|_A = h(T_\infty - T_A)$$

Note, we didn't look at this before since we usually dealt with low speed flows.

Example Space shuttle re-entry. Shuttle re-enters at up to $Ma = 25$. By the time it reaches lower altitude, it decreases down to $Ma \approx 5$. What is T_A ?

$$Ma = 5 \Rightarrow V = 5 \text{ (Speed of sound)} = 5(343 \text{ m/s})$$

$V = 1715 \text{ m/s} \Rightarrow$ Note, here we are assuming compressibility is not important.

$$T_A = T_\infty + \frac{V^2}{2C_p}; \quad C_p, \text{Air} \approx 1000 \text{ J/kg}\cdot\text{K}$$

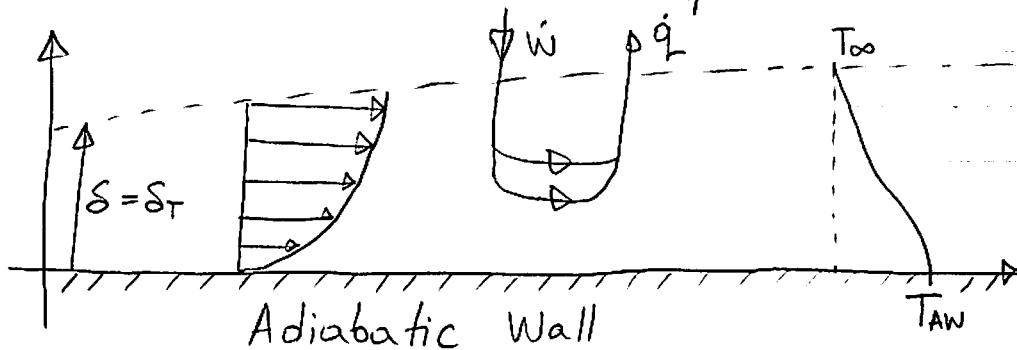
$$T_A = -40^\circ\text{C} + \frac{(1715 \text{ m/s})^2}{2(1000)}$$

$$\therefore \boxed{T_A \approx 1430^\circ\text{C}} \Rightarrow \text{Pretty hot!}$$

Note, this is true for any fast moving aircraft or projectile. Must be designed with thermal considerations in mind.

This solution is actually ok for stagnation points, but it's difficult to extend to flat plates.

For high velocity b.l. problems, velocity gradients within the b.l. cause mechanical to thermal energy conversion via viscous shear. Let's consider a simple case:



Note, the shear-to-thermal and kinetic-to-thermal energy conversion processes are fundamentally different. The shear mechanism is an irreversible process. Usually, most real cases involve both.

For shear on a flat plate, all the energy generated must escape by molecular or eddy conduction. At steady state, the wall will reach an adiabatic wall temperature, T_{aw} .

This implies that \Pr should be important in determining T_{aw} . We expect:

$\Pr \uparrow, T_{aw} \uparrow$ due to high ν , low α .

$\Pr \downarrow, T_{aw} \downarrow$ due to low ν , high α .

Let's do the math: Remember way back on pg. 14

$$\underbrace{p \frac{\partial e}{\partial t}}_{=0 \text{ Steady}} + e \left(\underbrace{\frac{\partial p}{\partial t} + p \nabla \cdot \mathbf{v}}_{=0 \text{ (Incompressible)}} \right) = -\nabla \cdot \mathbf{q}'' + q''' - \underbrace{\rho \nabla \cdot \mathbf{v}}_{=0 \text{ (no heat gen.)}} + \mu \overline{\Phi}$$

Assuming 2D, Incompressible, we simplified our energy eqn. to:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \overline{\Phi}$$

$$\overline{\Phi} = 2 \left(\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 \right) + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right)^2 \Rightarrow 20$$

Let's compare our terms with scaling:

$$\left. \begin{array}{l} U \sim U_\infty \\ x \sim L \\ V \sim U_\infty \left(\frac{s}{L} \right) \\ y \sim s \end{array} \right\} \quad \left. \begin{array}{l} \frac{\partial U}{\partial x} \sim \frac{U_\infty}{L} ; \quad \frac{\partial U}{\partial y} \sim \frac{U_\infty}{s} \\ \frac{\partial V}{\partial y} \sim U_\infty \frac{s}{L} ; \quad \frac{\partial V}{\partial x} \sim U_\infty \frac{s}{L^2} \end{array} \right.$$

$$\left. \begin{aligned} \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} &\sim \frac{\frac{U_\infty}{L}}{\frac{U_\infty}{S}} \sim \frac{S}{L} \ll 1 \Rightarrow \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y} \\ \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} &\sim \frac{\frac{U_\infty}{L}}{\frac{U_\infty}{S}} \sim \frac{S}{L} \ll 1 \Rightarrow \frac{\partial v}{\partial y} \ll \frac{\partial u}{\partial y} \\ \frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y}} &\sim \frac{\frac{U_\infty S}{L^2}}{\frac{U_\infty}{S}} \sim \left(\frac{S}{L}\right)^2 \ll 1 \Rightarrow \frac{\partial v}{\partial x} \ll \frac{\partial u}{\partial y} \end{aligned} \right\} \text{So } \frac{\partial u}{\partial y} \text{ dominates.}$$

∴ $\boxed{\Phi = \left(\frac{\partial u}{\partial y}\right)^2} \Rightarrow 2D \text{ flow,}$

So our energy equation becomes:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \underbrace{\frac{V}{C_p} \left(\frac{\partial u}{\partial y}\right)^2}_{\alpha Pr}$$

Now we'll go back to our familiar tricks (similarity)

$$f = f(r), \quad T = T(r)$$

$$U = \frac{\partial \Psi}{\partial Y}, \quad V = -\frac{\partial \Psi}{\partial X}, \quad \Psi = \sqrt{U \times U_\infty} f \quad \Rightarrow \text{see pg. 64 of notes for derivation.}$$

$$r = Y \sqrt{\frac{U_\infty}{U X}}$$

$$\text{We defined } f' = \frac{U}{U_\infty} \Rightarrow U = U_\infty f'$$

Let's do each term one by one:

$$U \frac{\partial T}{\partial x} = (U_\infty f') \frac{\partial T}{\partial r} \cdot \underbrace{\frac{\partial r}{\partial x}}_{\frac{\partial r}{\partial x}} = (U_\infty f') T' \left(-\frac{r}{2x}\right) = -\frac{U_\infty r}{2x} f' T'$$

(pg. 54 of notes)

$$\sqrt{\frac{\partial T}{\partial y}} = \frac{U_\infty}{2x} \sqrt{\frac{xy}{U_\infty}} \left(\eta f' - f \right) \cdot \frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \sqrt{\frac{U_\infty}{Ux}}$$

$$\sqrt{\frac{\partial T}{\partial y}} = \frac{U_\infty}{2x} \left(\eta f' T' - f T' \right)$$

Now for the right hand side:

$$\alpha \frac{\partial^2 T}{\partial y^2} = \alpha \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = \alpha \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) = \alpha \frac{\partial \eta}{\partial y} \left(\frac{\partial}{\partial \eta} \left(\frac{\partial T}{\partial \eta} \right) \cdot \frac{\partial \eta}{\partial y} \right)$$

$$\alpha \frac{\partial^2 T}{\partial y^2} = \alpha \left(\frac{\partial \eta}{\partial y} \right)^2 \cdot T'' = \frac{U_\infty \alpha}{Ux} T''$$

And the viscous term:

$$\frac{V}{C_p} \left(\frac{\partial U}{\partial y} \right)^2 = \frac{V}{C_p} \cdot \left(\frac{\partial U}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right)^2 = \frac{V}{C_p} \left(U_\infty f'' \cdot \sqrt{\frac{U_\infty}{Ux}} \right)^2$$

$$\frac{V}{C_p} \left(\frac{\partial U}{\partial y} \right)^2 = \frac{V U_\infty^3}{C_p X} (f'')^2$$

Putting it all together:

$$-\cancel{\frac{U_\infty \eta}{2x} f' T'} + \cancel{\frac{U_\infty \eta}{2x} f' T'} - \cancel{\frac{U_\infty}{2x} f T'} = \frac{U_\infty \alpha}{Ux} T'' + \frac{U_\infty^3}{C_p X} (f'')^2$$

$$\frac{1}{Pr} T'' + \frac{U_\infty^2}{C_p} (f'')^2 + \frac{1}{2} f T' = 0$$

$$T'' + \frac{U_\infty^2 Pr}{C_p} (f'')^2 + \frac{1}{2} Pr f T' = 0 \quad ①$$

So now we need to solve equation ①. Note, we can see that the equation is linear and that we have already solved for f before (Blasius).

$$\text{Let } \Theta(\eta) = \frac{T - T_\infty}{U_\infty^2 / 2 C_p} \quad ②$$

Substituting ② into ①

$$\Theta'' + \frac{1}{2} Pr f \Theta' + 2 Pr (f'')^2 = 0 \quad \boxed{③} \quad \Rightarrow \text{Second order, linear.}$$

Using Duhamel's theorem, we can solve this by superposition of a particular and homogeneous solution.

$$\Theta(n) = \Theta_p(n) + \Theta_c(n)$$

For the particular solution; let's assume adiabatic wall conditions: \Rightarrow We can do this due to Duhamel's theorem (Linear)

$$\Theta_p(n) \Rightarrow \left. \frac{\partial \Theta_p}{\partial n} \right|_0 = 0, \quad \Theta_p(n \rightarrow \infty) = 0$$

To solve we can use the integrating factor method or solve numerically using a shooting scheme:

$$\Theta_p(n=0) = \Theta_{AW}(n=0) = \int_0^\infty \frac{\int_0^n \exp\left(\int_0^2 \frac{1}{2} Pr f d\eta\right) 2 Pr (f'')^{1/2} d\eta}{\exp\left(\int_0^2 \frac{1}{2} Pr f d\eta\right)} d\eta$$

We can numerically integrate the above solution for various Pr and using $f(n)$ solution from Blasius (Tabulated result)

$$\Theta_p(0) = \begin{cases} Pr^{1/2}; & 0.5 \leq Pr \leq 47 \\ 1.9 Pr^{1/3}; & Pr \geq 47 \end{cases} \quad \begin{array}{l} \Rightarrow \text{Numerical solution} \\ \Rightarrow \text{Note, only solution at wall. Not full solution.} \end{array}$$

So for the adiabatic wall:

$$\Theta_p(0) = \Theta_{AW}(0) = \frac{T_{AW} - T_\infty}{\frac{1}{2} U_\infty^2 / C_p} = r \equiv \text{recovery factor}$$

$$T_{AW} = T_\infty + r \frac{U_\infty^2}{2C_p}, \quad r = \begin{cases} Pr^{1/2}; & 0.5 \leq Pr \leq 47 \\ 1.9 Pr^{1/3}; & Pr \geq 47 \end{cases}$$

So the "recovery factor" represents the fraction of kinetic energy "recovered" by an adiabatic wall.

Note, let's check the limit of the solution: for $\Pr = 1$

$$r = 1 \Rightarrow T_{AW} = T_\infty + \frac{U_\infty^2}{2C} \Rightarrow \text{Same as isentropic solution (stagnation point).}$$

Now we need the homogeneous solution

$$\Theta_H(n) \Rightarrow \Theta_H(n=0) = \text{constant (constant wall temp.)}$$

$$\Theta_H(n \rightarrow \infty) = 0 ; \quad \Theta_H'' + \frac{1}{2} \Pr f \Theta_H' = 0 \quad (4)$$

$$\Theta_H = \frac{T - T_0}{T_\infty - T_0} \Rightarrow \text{pg. } 55 \quad \hookrightarrow \text{Homogeneous Equation}$$

So:

$$\Theta(n) = \Theta_p + C_1 \Theta_H + C_2 \Rightarrow \text{Back substitute into (3) to check.}$$

Aside: Remember the following, for a nonhomogeneous linear ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y = y_c + y_p$$

Where y_p = any particular (specific) solution that satisfies the nonhomogeneous equation

$y_H = C_1 y_1 + C_2 y_2$ is the general solution to the homogeneous equation: (complementary sol'n)
 y_1 & y_2 are linearly independent solutions.

$$y'' + p(t)y' + q(t)y = 0 \Rightarrow \text{Homogeneous eqn.}$$

Back to our problem, we can now solve for C_1 & C_2 , using our overall b.c.s

Just to be sure though, we can back substitute our assumed solution into ③ and see if it makes sense:

$$\Theta'' + \frac{1}{2} Pr f \Theta' + 2 Pr (f'')^2 = 0$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial}{\partial \eta} (\Theta_p + C_1 \Theta_H + C_2) \right) + \frac{1}{2} Pr f \frac{\partial}{\partial \eta} (\Theta_p + C_1 \Theta_H + C_2) + 2 Pr (f'')^2 = 0$$

$$\underbrace{\Theta_p'' + \frac{1}{2} Pr f \Theta_p' + 2 Pr (f'')^2}_{=0 \text{ (Complementary solution)}} + \underbrace{C_1 \Theta_H'' + \frac{1}{2} Pr f C_1 \Theta_H'}_{=0 \text{ (Homogeneous solution)}} = 0$$

Now back to our B.C.'s

$$\Theta(\eta \rightarrow \infty) = 0 \quad \text{or} \quad T(\eta \rightarrow \infty) = T_\infty$$

$$\Theta_H(\eta \rightarrow \infty) = \frac{T_\infty - T_0}{T_\infty - T_0} = 1$$

$$\Theta_p(\eta \rightarrow \infty) = 0 \quad (\text{From numerical solution})$$

$$0 = 0 + C_1(1) + C_2 \Rightarrow C_1 = -C_2$$

Our second b.c. is: $\Theta(\eta=0)$

$$\Theta_H(\eta=0) = \frac{T_0 - T_\infty}{T_\infty - T_0} = 0$$

$$\Theta_p(\eta=0) = \Theta_{AW}(\eta=0) \quad (\text{Just a name change}) \quad \text{or } \Theta_p(0)$$

$$\frac{T_0 - T_\infty}{U_\infty^2 / 2 C_p} = \Theta_{AW}(0) + 0 + C_2 \Rightarrow C_2 = \frac{T_0 - T_\infty}{U_\infty^2 / 2 C_p} - \Theta_{AW}(0)$$

Back substituting everything into our complete solution

$$\Theta = \frac{T - T_\infty}{U_\infty^2 / 2 C_p} = \Theta_p + \left(\frac{T_0 - T_\infty}{U_\infty^2 / 2 C_p} - \frac{T_{0, AW} - T_\infty}{U_\infty^2 / 2 C_p} \right) (1 - \Theta_H) \quad \text{⑪0}$$

Rearranging, we obtain:

$$T - T_\infty = \Theta_p \frac{U_\infty^2}{2C_p} + (T_0 - \tilde{T}_{AW})(1 - \Theta_H)$$

Now we can evaluate the heat flux (note, Θ_p still is solved numerically). Θ_H we solved before (pg. 55 of notes).

$$q''|_0 = -k \left. \frac{\partial T}{\partial y} \right|_0$$

$$\left. \frac{\partial T}{\partial y} \right|_0 = \left. \frac{\partial \Theta_p}{\partial y} \right|_0 \cdot \frac{U_\infty^2}{2C_p} - (T_s - T_{AW}) \left. \frac{\partial \Theta}{\partial y} \right|_0 ; \text{ note } \left. \frac{\partial \Theta_p}{\partial y} \right|_0 = 0 \text{ (Adiabatic)}$$

$$\left. \frac{\partial T}{\partial y} \right|_0 = 0 - (T_s - T_{AW}) \left. \frac{\partial \Theta}{\partial y} \right|_0$$

$$q''|_0 = -k \left. \frac{\partial T}{\partial y} \right|_s = k (T_s - T_{AW}) \frac{\partial \Theta_H}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$q''|_0 = k (T_s - T_{AW}) \left. \frac{\partial \Theta_H}{\partial \eta} \right|_0 \cdot \sqrt{\frac{U_\infty}{Vx}}$$

But we've already solved for $\Theta_H|_0$ on pg. 56

$$\Theta(\eta^*) = f'(\eta)$$

$$\frac{\partial \Theta_H}{\partial \eta} = \frac{\partial \Theta_H}{\partial \eta^*} \cdot \frac{\partial \eta^*}{\partial \eta} \Rightarrow \eta^* = \eta \Pr^{1/3} \Rightarrow \frac{\partial \eta^*}{\partial \eta} = \Pr^{1/3}$$

$$\left. \frac{\partial \Theta_H}{\partial \eta} \right|_0 = \left. \frac{\partial \Theta_H}{\partial \eta^*} \right|_0 \cdot \Pr^{1/3} = f''(0) \cdot \Pr^{1/3} = 0.332 \Pr^{1/3}$$

$$\therefore \boxed{\left. \frac{\partial \Theta_H}{\partial \eta} \right|_0 = 0.332 \Pr^{1/3}} \Rightarrow \text{Back substitute into } q''|_0$$

$$\boxed{q''|_0 = k (T_0 - T_{AW}) \frac{0.332 \Pr^{1/3}}{\sqrt{Vx/U_\infty}}} \Rightarrow \text{Same result as before}$$

but $T_\infty = T_{AW}$

So we can say for flows with viscous dissipation

So for viscous heating b.l.'s :

$$\left. q'' \right|_0 = h (T_0 - T_{\infty})$$

$$h = k \frac{0.332 \Pr^{1/3}}{\sqrt{U_x/U_\infty}} \Rightarrow \text{Same as before}$$

$$Nu_x = \frac{h_x}{k} = 0.332 \Pr^{1/3} Re_x^{1/2} \Rightarrow \text{Same as before}$$

The only change is instead of using T_∞ , we should use T_{∞} . Very cool!

Let's test the limits of this:

$$T_{\infty} = T_\infty + r \frac{U_\infty^2}{2C_p}, \quad r = \begin{cases} \Pr^{1/2}; & 0.5 \leq \Pr < 47 \\ 1.9 \Pr^{1/3}; & \Pr \geq 47 \end{cases}$$

For $\Pr = 1$, $r = 1$ so

$$T_{\infty} = T_\infty + \frac{U_\infty^2}{2C_p} \Rightarrow \text{Same as previous solution (stagnation pt.)}$$

$$\text{For: } \left. q'' \right|_0 = h \left(T_0 - T_\infty - \frac{U_\infty^2}{2C_p} \right) \Rightarrow \frac{U_\infty^2}{2C_p(T_0 - T_\infty)} \ll 1$$

Then $T_{\infty} = T_\infty$ and our old results are valid.

Here we see why we defined the Eckert number like we did:

$$Ec = \frac{U_\infty^2}{C_p(T_0 - T_\infty)}$$

\Rightarrow It's nothing but a measure of the negligibility of the recovery factor term.

Note to be more rigorous, we should say:

$$Ec = \frac{r U_\infty^2}{2C_p(T_0 - T_\infty)} \ll 1 \text{ for } T_{\infty} = T_\infty \text{ since } r = f(\Pr)$$

Sometimes our condition is written as $\Pr \cdot Ec \ll 1$

So how about our second condition (temperature dependent properties). Well, Eckert found a miraculous solution. As long as we take our properties at: (as long as $C_p \approx \text{const}$)

$$T_R = T_\infty + 0.5(T_0 - T_\infty) + 0.22(T_{Aw} - T_\infty) \quad \begin{matrix} \text{for } Ma < 20 \\ \text{Pr} < 15 \end{matrix}$$

All properties are evaluated at T_R , including Pr, p, μ, k . Note, this also takes into account compressibility effects.

This also works for calculating shear stress & local skin friction coefficient. Use T_R as the reference temperature.

Example #1 For turbulent flows, it's been shown that:

$$r \approx \text{Pr}^{1/3} \quad \text{for all Pr, and } 0 < Ma < 8, \text{ and gas flow}$$

Find T_{Aw} for the space shuttle re-entry. ($Ma = 5$)

$$V = 5(343 \text{ m/s}) = 1715 \text{ m/s}$$

$$C_p, \text{Air} \approx 1000 \text{ J/kg}\cdot\text{K}$$

$$\text{Pr, Air} \approx 0.68 \quad (\text{at } T = 400^\circ\text{C})$$

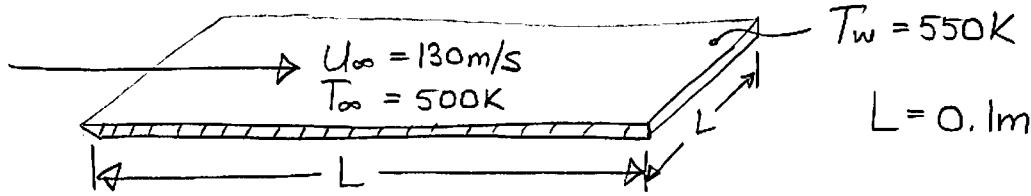
$$r = \text{Pr}^{1/3} = 0.88 \Rightarrow T_{Aw} = -40^\circ\text{C} + (0.88) \frac{(1715 \text{ m/s})^2}{2(1000 \text{ J/kg}\cdot\text{K})}$$

$$T_{Aw} \approx 1254^\circ\text{C}$$

Note, now we should iterate to get properties at T_R . C_p won't change that much, but Pr will.

Also, at very high gas velocities ($Ma > 20$), the wall gets so hot that ionization of the gas occurs, which needs to be taken into account. This ionization is why we lose radio communication with re-entry vehicles. The charged ions act as a blocking mechanism for EM waves.

Example #2] Flow past a flat plate (air)



First let's check to see if viscous heating is important:

$$\Pr \cdot E_C = \Pr \frac{U_\infty^2}{C_p \cdot \Delta T} = 0.23 \Rightarrow \text{Need to consider}$$

$$Re_L = 310\,000 \approx 5.0 \times 10^5 \quad (\text{Laminar})$$

$$Ma = 0.28 \approx \text{incompressible}$$

Now we can solve since all the conditions fit our developed correlations.

$$T_{Aw} = T_\infty + \frac{r U_\infty^2}{2 C_p}, \quad r = \Pr^{1/2} = (0.68)^{1/2} = 0.82$$

$$T_{Aw} = 500\text{ K} + \frac{0.82(130\text{ m/s})^2}{2(1000\text{ J/kg}\cdot\text{K})} = 506.9\text{ K}$$

$$T_R = T_\infty + 0.5(T_o - T_\infty) + 0.22(T_{Aw} - T_\infty) \\ = 500\text{ K} + 25\text{ K} + 0.22(6.9\text{ K})$$

$$T_R = 526.5\text{ K} \Rightarrow \rho_{\text{Air}} = 0.65\text{ kg/m}^3, \quad C_{p,\text{Air}} \approx 1050\text{ J/kg}\cdot\text{K} \\ \mu_{\text{Air}} = 2.67 \times 10^{-5}\text{ kg/m}\cdot\text{s}, \quad \Pr \approx 0.68, \quad k_{\text{Air}} = 0.04\text{ W/m}\cdot\text{K}$$

$$q''|_o = h L^2 (T_o - T_{Aw})$$

$$h = \overline{Nu}_L \cdot \frac{k_{\text{Air}}}{L} = \frac{k_{\text{Air}}}{L} \cdot (0.664 Re_L^{1/2} \Pr^{1/3}) = 136.3\text{ W/m}^2\cdot\text{K}$$

$$\boxed{q''|_o = 58.84\text{ W}} \quad \text{heat}$$

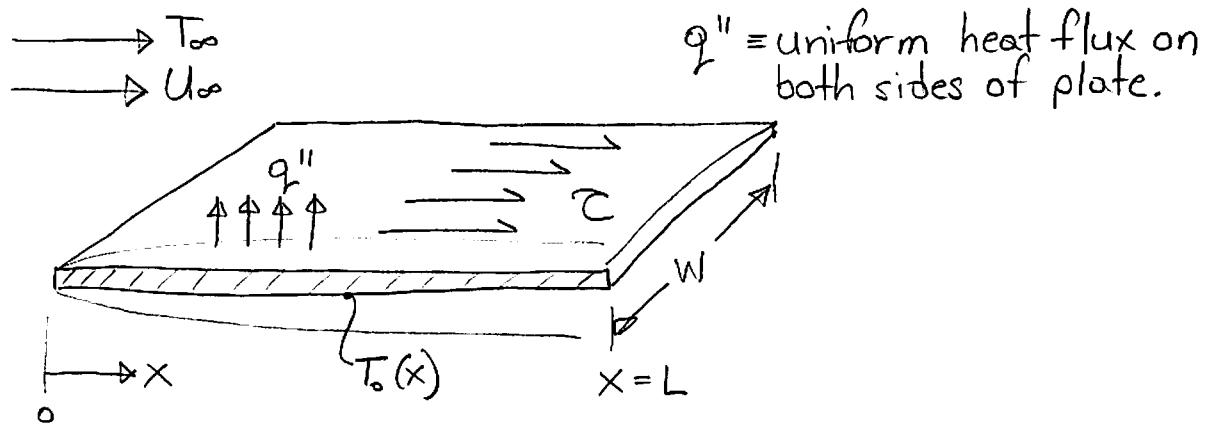
$$\boxed{q''|_{o,\text{false}} = h A (T_o - T_\infty) = 68\text{ W}}$$

$q''^{\text{real}} < q''^{\text{false due}}$

Entropy Generation Minimization in External Laminar B.L. Flow

We now have an excellent understanding of heat transfer and fluid flow (friction) in laminar b.l.'s. So how can we now design them such that the entropy generated during the process is minimal (i.e. what geometry will result in $S_{gen,min}$?).

Consider the following case:



We know from the first part of the class that for any point in the flow field:

$$S''_{gen} = \underbrace{\frac{k}{T^2} (\nabla T)^2}_{\geq 0} + \underbrace{\frac{4}{T} \frac{d}{dx}}_{\geq 0} \geq 0$$

If we volumetrically integrate this expression over the entire fluid domain, we obtain a fundamental result for entropy generation due to heat transfer between a body and a flow (U_∞, T_∞) surrounding the body:

$$S_{gen} = \underbrace{\frac{1}{T_\infty^2} \int_A q'' (T_0 - T_\infty) dA}_{\text{Heat Transfer}} + \underbrace{\frac{F_0 U_\infty}{T_\infty}}_{\text{Friction (Shear)}}$$

Note: $T_\infty \gg (T_0 - T_\infty)$;

F_0 = Drag force on the body ; $q''(T_0 - T_\infty) \geq 0$ always
 A = Body surface area

So let's solve our problem: We can already intuitively tell that it will involve optimization since

$$\begin{aligned} q''(T_0 - T_\infty) \downarrow & \text{ as } L \uparrow \Rightarrow (\overline{T}_0 - T_\infty) \downarrow \text{ or } q'' \downarrow \\ F_0 \uparrow & \text{ as } L \uparrow \end{aligned} \quad \begin{array}{l} \text{For constant } q'' \\ \text{For constant } (\overline{T}_0 - T_\infty) \end{array}$$

We can simplify our integral to the following:

$$\int q''(T_0 - T_\infty) dA = q''(\overline{T}_0 - T_\infty)(2LW)$$

We can solve for $(\overline{T}_0 - T_\infty)$ since we already solved the constant heat flux case

$$Nu_x = \frac{q''|_0}{T_0(x) - T_\infty} \cdot \frac{x}{k} = 0.455 Pr^{1/3} Re_x^{1/2} \quad (0.5 \leq Pr \leq 10)$$

From pg. 102 of notes.

Rearranging:

$$\begin{aligned} (\overline{T} - T_\infty) &= \frac{1}{L} \int_0^L \frac{q'' \cdot x}{0.455 Pr^{1/3} Re_x^{1/2} k} dx \\ &= \frac{q'' u^{1/2}}{0.455 \rho^{1/2} U_\infty^{1/2} k Pr^{1/3} L} \int_0^L x^{1/2} dx \\ &= \frac{q'' L}{0.455 \rho^{1/2} U_\infty^{1/2} L^{1/2} \cdot k Pr^{1/3} u^{1/2}} \cdot \frac{2}{3} = \frac{q'' L}{0.455 Pr^{1/3} Re_L^{1/2} k} \left(\frac{2}{3}\right) \end{aligned}$$

So now we can compute our entropy due to heat transfer:

$$\int q''(T_0 - T_\infty) dA = q''(\overline{T} - T_\infty)(2LW) = \frac{0.732(q')^2 \cdot W}{k Re_L^{1/2} Pr^{1/3} T_\infty^2} \quad ①$$

Where we've defined $q' = 2Lq''$ \Rightarrow Heat transfer rate per unit depth (w-direction).

Now we can deal with the shear term:

$$\frac{F_0 U_\infty}{T_\infty} = \frac{(2LW) \bar{C} U_\infty}{T_\infty} \Rightarrow \text{For a flat plate in laminar flow: } \bar{C} = 0.664 \rho U_\infty^2 Re_L^{1/2}$$

$$= \frac{1.328 \rho U_\infty^3 w L}{Re_L^{1/2}} = \frac{1.328 \rho^{1/2} U_\infty^{5/2} w L^{1/2} u}{u^{1/2} T_\infty}$$

So:

$$\frac{F_0 U_\infty}{T_\infty} = \frac{1.328 Re_L^{1/2} U_\infty^2 W \cdot u}{T_\infty} \quad (2)$$

Putting ① & ② together:

$$\frac{S_{gen}}{W} = \frac{S_{gen, HT}}{W} + \frac{S_{gen, SHEAR}}{W} = \frac{0.736 (q')^2}{T_\infty^2 k \Pr^{1/3} Re_L^{1/2}} + 1.328 \frac{u U_\infty^2 Re_L^{1/2}}{T_\infty}$$

So now we can optimize our plate design.

The $Re_L^{1/2}$ term appears in both terms. We can differentiate and solve for $L_{optimum}$.

$$\frac{\partial S_{gen}}{\partial Re_L} = 0 \Rightarrow \text{Solve for } Re_{L, \text{opt}}$$

$$Re_{L, \text{opt}} = 0.554 B^2$$

$$B = \frac{q'}{U_\infty (k u T_\infty \Pr^{1/3})^{1/2}} = \text{Bejan \#}$$

$$B = \frac{\text{Heat transfer rate}}{\text{Flow speed}} \quad (\text{Dimensionless})$$

→ Governs the entropy generation characteristics in laminar boundary layer flows.

If: $Re_L \ll B^2 \Rightarrow$ Heat transfer dominates entropy generation

$Re_L \gg B^2 \Rightarrow$ Fluid friction dominates entropy generation

In conclusion, if a plate (fin) is to transfer a constant heat transfer per unit depth ($q' = \text{const.}$) to a stream with constant U_{∞} and T_{∞} , then:

$$L_{\text{opt}} = 0.554 \frac{(q')^2}{k T_{\infty} \rho U_{\infty}^3 Pr^{1/3}}$$

\Rightarrow Optimum length for min. entropy generation.

$$S_{\text{gen, min}} = 1.98 \frac{q' U_{\infty}}{(k/\mu)^{1/2} T_{\infty}^{3/2} Pr^{1/6}}$$

$\Rightarrow S_{\text{gen}}$ for $L = L_{\text{opt}}$.

Note, the above analysis is fine and dandy but see if you can spot the one key assumption.

Answer: We've assumed in our solution that:

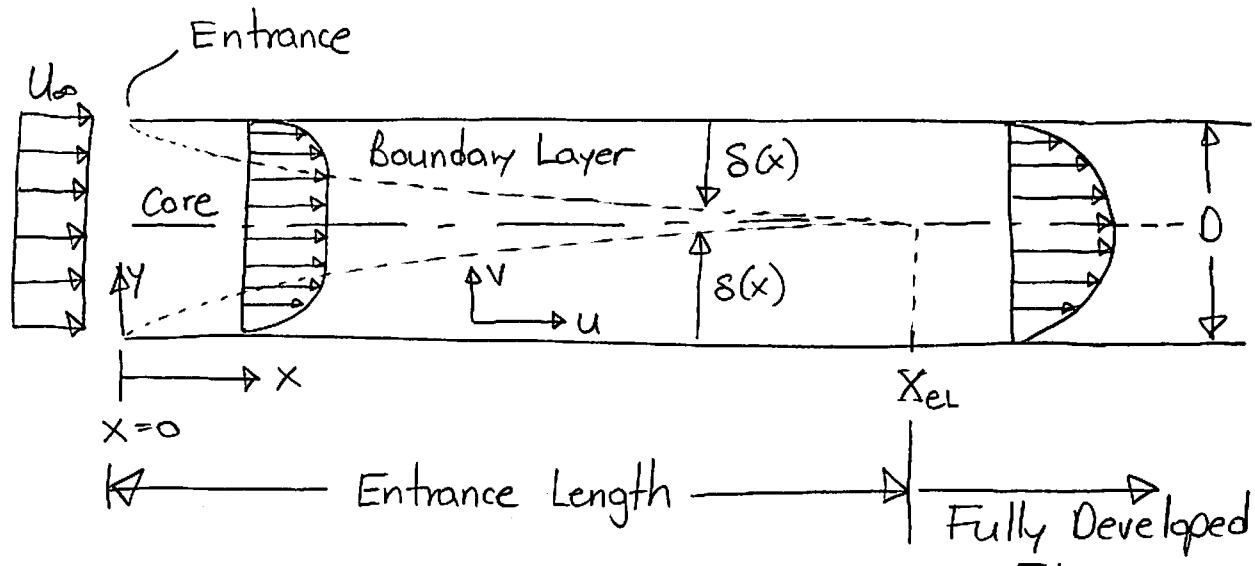
$$q' = \text{constant} = 2q''L$$

OK, but that means in order for each fin length we analyze, q'' is changing. If we wanted an optimization with $q'' = \text{constant}$, things would get complicated.

This is the nice part about entropy generation minimization. You can design heat transfer components in ways not possible before.

Internal Flow

Consider 2 parallel plates forming a 2D duct.



Let's estimate the entrance length (X_{el}).

We expect here that:

$$\text{For: } x < X_{el} \Rightarrow \overline{h_x} \uparrow, \overline{C_x} \uparrow$$

$$x > X_{el} \Rightarrow \overline{h_x} \downarrow, \overline{C_x} \downarrow$$

To estimate X_{el} , we can use scaling. We know from previous derivation that for a flat plate laminar flow:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \quad (\text{Blasius solution, pg. 51 of notes})$$

When the two boundary layers merge: $\delta = \frac{D}{2}, x = X_{el}$

$$\begin{aligned} \frac{D}{2X_{el}} &= \frac{5U^{1/2}}{\rho^{1/2} U_{in}^{1/2} X_{el}^{1/2}} \Rightarrow \frac{D}{X_{el}} = \frac{10U^{1/2}}{\rho^{1/2} U_{in}^{1/2} X_{el}^{1/2}} \cdot \frac{D^{1/2}}{D^{1/2}} \\ &= \frac{10}{\rho^{1/2} U_{in}^{1/2} D^{1/2}} \cdot \left(\frac{D}{X_{el}}\right)^{1/2} \end{aligned}$$

$$\text{So: } \left(\frac{D}{X_{el}}\right)^{1/2} = \frac{10}{Re_0} \quad \text{or:}$$

$$\boxed{\frac{X_{el}}{D} = 0.01 Re_0}$$

But note, this is only an estimate as our core flow is not $U_\infty = \text{constant}$. Here, the core is being squeezed and accelerates with the flow. (i.e. $U_\infty = f(x)$).

For a more accurate estimate, we can use integral methods to solve for X_{cr} . From pg. 82 of our notes: (momentum integral eqn).

$$\frac{2}{\rho} \int_0^Y u(U_\infty - u) dy = \frac{1}{\rho} \frac{\partial P_\infty}{\partial x} \cdot Y + U \frac{\partial U}{\partial Y} \Big|_0 - \frac{2U_\infty}{\rho} \int_0^Y u dy \quad (1)$$

Since our core flow is inviscid, we can apply Bernoulli's equation along the core streamline: ($U_c = U_\infty$ in the core).

$$\rho U_c^2 / 2 + P_\infty = \text{constant} \Rightarrow \text{Differentiate since we need } \frac{\partial P_\infty}{\partial x}$$

$$\frac{\partial}{\partial x} \left(\frac{2}{\rho} U_c^2 \right) + \frac{\partial P_\infty}{\partial x} = 0$$

$$U_c \frac{\partial U_c}{\partial x} + \frac{1}{\rho} \frac{\partial P_\infty}{\partial x} = 0 \quad (2)$$

$$\frac{\partial U_c}{\partial x} \int_0^S u_c dy = U_c \frac{\partial U_c}{\partial x} S$$

Back substituting (2) into (1):

$$\frac{2}{\rho} \left[\int_0^S (U_c - U) u dy + \frac{2U_c}{\rho} \int_0^S (U_c - U) dy \right] = U \frac{\partial U}{\partial Y} \Big|_0 \quad (3)$$

Note, I let $Y = S$, and only integrated up to the channel centerline due to symmetry.

To solve the integrals, we can first apply conservation of mass:

$$\boxed{\int_0^S \rho u dy + \int_S^{D/2} \rho U_c dy = \rho U_\infty \frac{D}{2}} \quad (4)$$

Now we can solve equations (3) & (4) by assuming a velocity profile. As before:

$$\frac{U}{U_c} = \frac{2y}{S} - \left(\frac{y}{S} \right)^2$$

Back substituting and solving, we obtain:

$$\frac{x}{D \cdot Re_0} = \frac{3}{40} \left(9 \frac{U_c}{U_\infty} - 2 - 7 \frac{U_\infty}{U_c} - 16 \ln \left(\frac{U_c}{U_\infty} \right) \right) \quad (5)$$

and:

$$\frac{2S}{D} = 3 \left[1 - \frac{U_\infty}{U_c} \right] \quad (6) \Rightarrow \text{Note } S = f(x) \text{ and } U_c = f(x)$$

When our two b.l.'s merge, $S(X_{el}) = \frac{D}{2}$

$$\frac{20}{20} = 3 \left[1 - \frac{U_\infty}{U_c} \right] \Rightarrow U_c(X_{el}) = \frac{3}{2} U_\infty \quad (7)$$

Back substitute (7) into (5) and solve for X_{el}/D :

$$\frac{X_{el}}{D} = 0.026 Re_0$$

\Rightarrow More accurate solution.

Experiments show that:

$$\frac{X_{el}}{D} \approx 0.04 Re_0 \text{ to } 0.05 Re_0$$

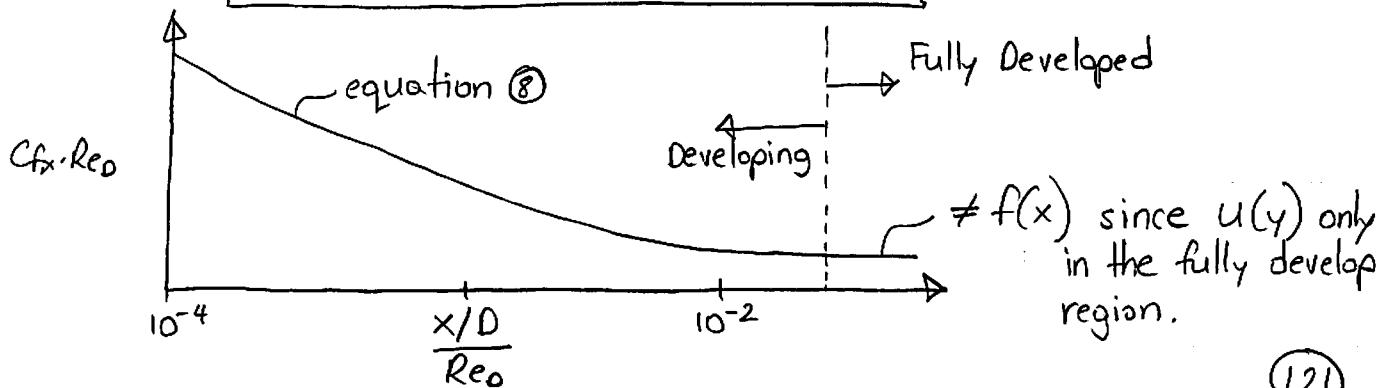
Minor difference is due to edge effects where the b.l. analysis breaks down.

Fundamental differences between the entrance and fully developed region is the wall shear stress: $C_{fx} = C_x / \frac{1}{2} \rho U_\infty^2$

Defining $C_x = U \frac{\partial U}{\partial y}|_0$ and using our solution (eqn's (5) & (6))

$$C_{fx} \cdot Re_0 = \frac{8}{3} \frac{U_c}{U_\infty} \left(1 - \frac{U_\infty}{U_c} \right)^{-1} \quad (8)$$

\Rightarrow Plot below:



Fully-Developed Flow

At any point in our channel, our mass & momentum conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1) \Rightarrow \text{mass conservation}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2) \Rightarrow x\text{-momentum}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3) \Rightarrow y\text{-momentum}$$

Using scaling for a fully developed flow:

$$\left. \begin{array}{l} x \sim L \\ y \sim D \\ u \sim U_\infty \end{array} \right\} \text{From eq. (1)} \quad \frac{U_\infty}{L} + \frac{v}{D} = 0 \Rightarrow v \sim \frac{DU_\infty}{L} \quad (4)$$

For fully developed flow, $x > L$ such that $v \sim \frac{DU_\infty}{L} \ll U_\infty$
 So far enough from the entrance such that
 the scale of v is negligible.

So for fully developed flow:

$$v = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0 \quad \text{so} \quad \frac{\partial u}{\partial x} = 0 \Rightarrow \text{From continuity}$$

Note in the entrance region ($x < X_{el}$), $y \sim \delta$ so $v \neq 0$
 Applying this to the y -momentum equation: (eq. (3))

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \Rightarrow v = 0 = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$\therefore \frac{\partial p}{\partial y} = 0 \Rightarrow p = f(x) \text{ only}$$

$$\frac{\partial p}{\partial x} = u \frac{\partial^2 u}{\partial y^2} \quad (5)$$

Looking at x -momentum (eq. (2)):

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \frac{\partial u}{\partial x} = 0, v = 0$$

Note, since $P=f(x)$ only and $U=f(y)$ only (i.e. $\frac{\partial U}{\partial x}=0$)

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 U}{\partial y^2} = \text{constant} \quad (6)$$

Now we can solve eq. (6) subject to our b.c.'s : (note $y=0$ -centerline)
 $U(y=\frac{D}{2}) = 0 \Rightarrow \text{no-slip}$
 $U(y=-\frac{D}{2}) = 0 \Rightarrow \text{no-slip}$

} Integrate (6) twice & apply these.

$$U = \frac{3}{2} \bar{U} \left[1 - \left(\frac{y}{D/2} \right)^2 \right] \Rightarrow \text{velocity profile, } u(y)$$

$$\bar{U} = \frac{D^2}{12\mu} \left(-\frac{\partial P}{\partial x} \right) \Rightarrow \text{Average velocity: } \bar{U} = \frac{1}{D} \int_{-\frac{D}{2}}^{+\frac{D}{2}} u dy$$

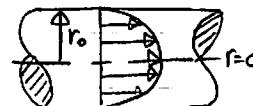
Here, y is the distance away from the centerline of the channel.

In general, we can say:

$$\frac{\partial P}{\partial x} = \mu \nabla^2 U = \text{constant}$$

Where ∇^2 = Laplacian operator.

For example, for a round tube with radius r_0 :



$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \Rightarrow \text{Note, I've neglected the } \theta \& z \text{ terms.}$$

$$\frac{\partial P}{\partial x} = \mu \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) \Rightarrow \text{Solving this with: } U(r=r_0)=0$$

$$\left. \frac{\partial U}{\partial r} \right|_{r_0} = 0$$

$$U = 2\bar{U} \left[1 - \left(\frac{r}{r_0} \right)^2 \right] \Rightarrow \text{velocity profile, } u(r)$$

$$\bar{U} = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x} \right) \Rightarrow \text{Average velocity: } \bar{U} = \frac{1}{\pi r_0^2} \int_0^{r_0} (2\pi r) U dr$$

$$m = \frac{\pi r_0^4}{8\mu} \left(-\frac{\partial P}{\partial x} \right) \Rightarrow \text{Mass flow rate [kg/s]: } m = \rho \bar{U} \pi r_0^2$$

These solutions were first reported by Hagen in 1839 and Poiseulle in 1840.

These pressure-driven flows are called Hagen-Poiseuille flows. We commonly assign a Reynolds number to these flows.

$$Re = \frac{UD}{V} = \frac{\text{inertial force}}{\text{viscous force}}$$

But when we started our analysis with equation ②

$$\underbrace{U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y}}_{\text{Inertial Forces}} = - \underbrace{\frac{1}{\rho} \frac{\partial P}{\partial x}}_{\text{Pressure Force}} + \underbrace{V \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)}_{\text{Viscous Forces}}$$

Since we showed for fully developed flow that: $V=0, \frac{\partial U}{\partial x}=0$

\therefore Inertial Forces = 0!

Indeed, looking at our governing equation for these flows:

$$\underbrace{\frac{\partial P}{\partial x}}_{\text{Pressure Forces}} = \underbrace{\nabla^2 U}_{\text{Viscous Forces}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{So the concept of Reynolds number in Hagen-Poiseuille flows is } \underline{\text{nonsense!}}$$

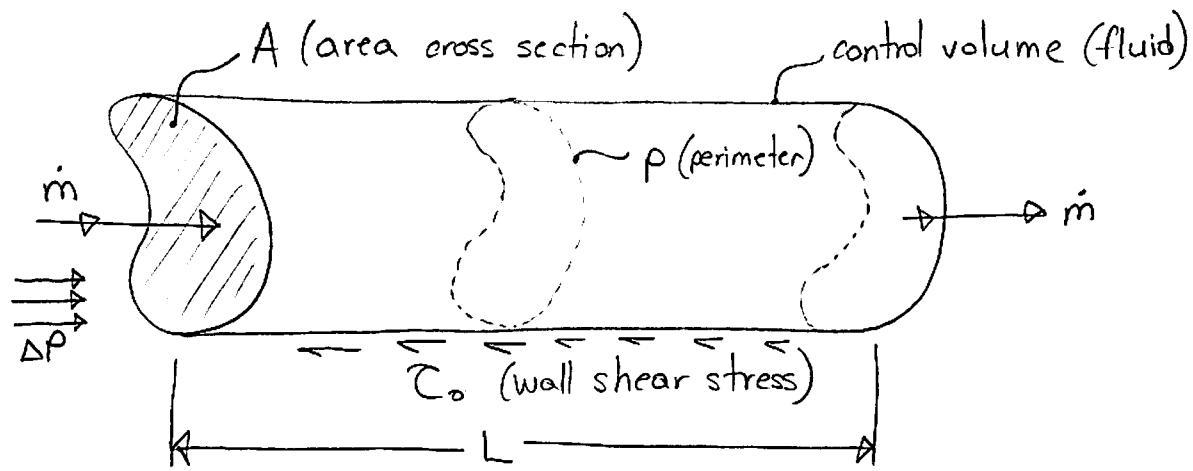
These flows are governed by Pressure \sim Viscosity, or

$$\frac{-\frac{\partial P}{\partial x}}{U \frac{\partial^2 U}{\partial x^2}} \sim \frac{\Delta P / L}{U \bar{U} / D^2} \sim O(1) = \frac{\text{longitudinal pressure force}}{\text{friction force}} \quad \xrightarrow{\text{On the order of:}}$$

Note, if you want to prove this to yourself, for a flow with $Re \approx 2000 \Rightarrow$ The inertia is 2000x larger than the friction force! So theoretically, if I turn off my pump which generates $\frac{\partial P}{\partial x}$, the flow should keep going for a long time. In reality, as soon as you turn off a pump, the channel flow will stop almost immediately.

Hydraulic Diameter and Pressure Drop

The methodology of obtaining the relationship between m vs. ΔP stems from hydraulic theory of the 18th century. This is still useful today especially for turbulent flows. For our case, we can obtain an analytical result:



For a fully developed flow, we can apply a force balance on our fluid CV:

$$\sum F_x = 0 \quad (\text{since fully developed, steady})$$

$$A\Delta P = \rho L \tau_0 \Rightarrow \text{Note, } p \equiv \text{wetted perimeter}$$

$$\boxed{\tau_0 = \frac{\Delta P}{\left(\frac{L}{A/p}\right)}}$$

Here, we can define a friction factor, f , as:

$$\boxed{f = \frac{\tau_0}{\frac{1}{2} \rho \bar{U}^2}}$$

Note, f is similar to c_f but for fully developed flow, $f \neq f(x) = \text{constant}$
Rearranging:

$$\boxed{\Delta P = f \frac{\rho L}{A} \left(\frac{1}{2} \rho \bar{U}^2\right)}$$

⇒ Pressure drop across the duct

Finally, note that A/p is the linear dimension of the cross section:

$$\boxed{r_h = \frac{A}{p} = \text{hydraulic radius}}$$

$$\boxed{D_h = 4r_h = \frac{4A}{P} = \text{hydraulic diameter}}$$

The physical meaning of D_h can be understood as the length that accounts for how close the wall (and its resistive forces - shear) are positioned relative to the stream.

We can now solve for f for our Hagen-Poiseulle flows:
Let's use the round tube for simplicity:

$$\bar{U} = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x} \right) \Rightarrow \frac{8\mu \bar{U}}{r_0^2} = \frac{\Delta P}{L}$$

$$\text{We know: } f = \frac{\Delta P}{\left(\frac{4L}{D_h}\right) \frac{1}{2} \rho \bar{U}^2} \Rightarrow D_h = \frac{4A}{P} = \frac{4(\pi r_0^2)}{2\mu \bar{U}} = 2r_0$$

$$f = \frac{\Delta P}{\left(\frac{4KL}{2r_0}\right) \frac{1}{2} \rho \bar{U}^2} \Rightarrow \left(\frac{\Delta P}{L}\right) \cdot \frac{0}{2\rho \bar{U}^2} = \frac{8\mu \bar{U}}{r_0^2} \cdot \frac{0}{2\rho \bar{U}^2}$$

$$f = \frac{4\mu D}{\left(\frac{D}{2}\right)^2 \cdot \rho \bar{U}} = \frac{16\mu}{\rho \bar{U} D} = \frac{16}{Re_{D_h}}$$

$$\boxed{f = \frac{16}{Re_{D_h}}}, \quad D_h = D \text{ (round tube)}, \quad \boxed{Re_D = \frac{\rho \bar{U} D}{\mu}}$$

For parallel plates, we can do a similar analysis to show:

$$\boxed{f = \frac{24}{Re_{D_h}}}, \quad D_h = 2D \quad (D = \text{gap thickness}).$$

Note, these are valid for $Re_{D_h} < 2000$. Also, note the convention. We have derived the "Fanning" friction factor. There is another convention that is used for pipes:

$$\boxed{f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho \bar{U}^2} = \frac{64}{Re_D} = \text{Darcy friction factor}}, \quad Re_D < 2300$$

Laminar flow
Round tube.

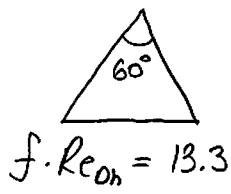
Typically, friction factor results are tabulated as $f \cdot Re_0 = \text{constant}$.

Interestingly, we can see from our previous scaling:

$$\frac{\Delta P/L}{\mu \bar{U}/D_h^2} = \frac{\frac{4f}{D_h} \left(\frac{1}{2} \rho \bar{U}^2 \right)}{\mu \bar{U}/D_h^2} = \frac{f \rho \bar{U} D_h}{\mu} = f \cdot Re_{D_h}$$

So $\frac{\Delta P/L}{\mu \bar{U}/D_h^2} = f \cdot Re_{D_h} \sim O(1)$ \Rightarrow Makes sense. As we expected.

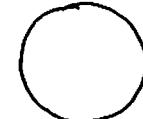
For many cross sections, $f \cdot Re_{D_h} \sim O(1)$ indeed.



$$f \cdot Re_{D_h} = 13.3$$



$$f \cdot Re_{D_h} = 14.2$$



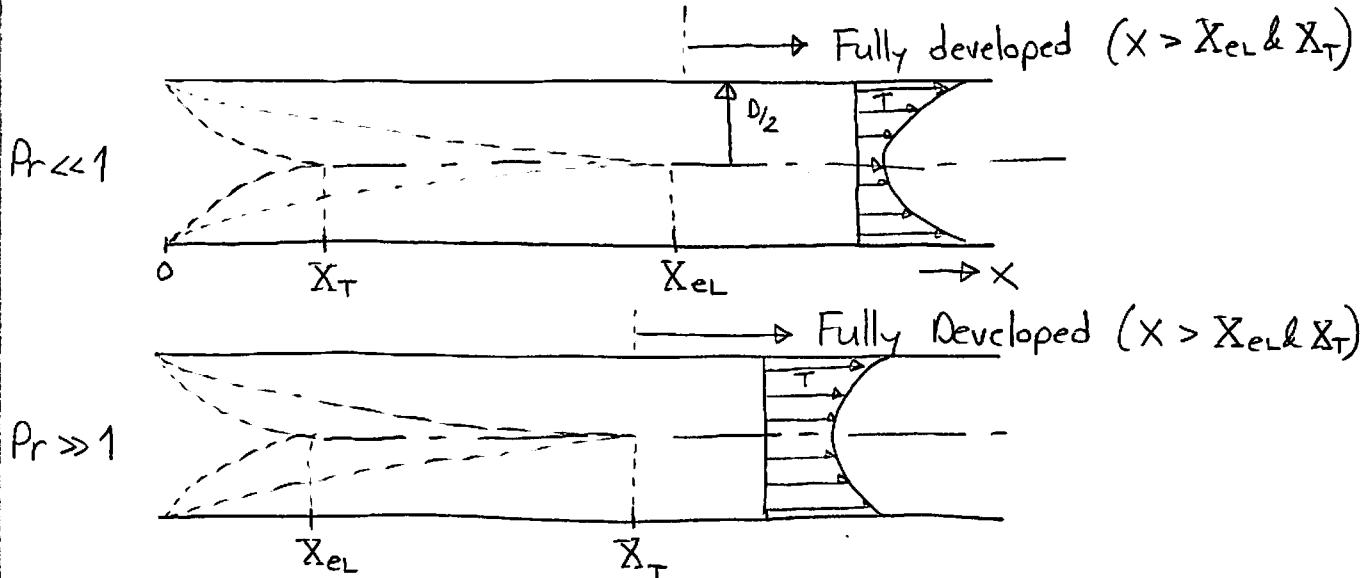
$$f \cdot Re_{D_h} = 16$$



$$f \cdot Re_{D_h} = 24$$

Heat Transfer in Fully Developed Duct Flow

To begin, we can first analyze when fully developed thermal conditions are met. We have 2 cases to consider:



For $\text{Pr} \ll 1$: $S_T \sim \propto \text{Pr}^{-\frac{1}{2}} \text{Re}_x^{-\frac{1}{2}}$ (pg. 42 of notes)

For our case, $x \sim X_T$ and $S_T \sim \frac{D_h}{2}$ (note, some books use $S_T \sim D_h$)

$$X_T \text{Pr}^{-\frac{1}{2}} \text{Re}_{x_T}^{-\frac{1}{2}} \sim \frac{D_h}{2} \Rightarrow \text{convert } \text{Re}_{x_T} \text{ to } \text{Re}_{D_h}$$

$$\text{Re}_{x_T} = \frac{\rho \bar{U} X_T}{\mu} \left(\frac{D_h}{D_h} \right) = \frac{\rho \bar{U} D_h}{\mu} \cdot \left(\frac{X_T}{D_h} \right)$$

$$X_T \text{Pr}^{-\frac{1}{2}} \text{Re}_{D_h}^{-\frac{1}{2}} \left(\frac{X_T}{D_h} \right)^{-\frac{1}{2}} \sim \frac{D_h}{2}$$

$$\boxed{\left(\frac{X_T / D_h}{\text{Pr} \cdot \text{Re}_{D_h}} \right)^{\frac{1}{2}} \sim \frac{1}{2}} \quad ①$$

\Rightarrow Thermal b.l. development

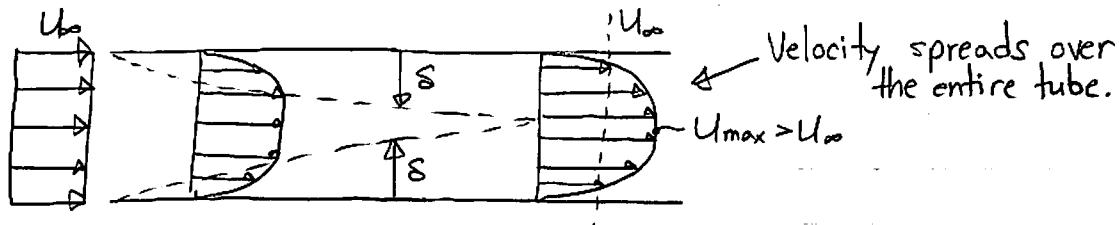
Note, we could have done the same thing for the hydrodynamic b.l.:

$$\boxed{\left(\frac{X_{el} / D_h}{\text{Re}_{D_h}} \right)^{\frac{1}{2}} \sim \frac{1}{2}} \quad ②$$

\Rightarrow Hydrodynamic b.l. development

For $\text{Pr} \gg 1$: It is tempting to say that $U \sim \left(\frac{S_T}{S} \right) U_\infty$ like we did before, inside a layer of thickness S_T . However, since our channel is confined, and the velocity scale spreads over D_h , hence $U \sim U_\infty$ and:

$$S_T \sim \propto \text{Pr}^{-\frac{1}{2}} \text{Re}_x^{-\frac{1}{2}} \Rightarrow \text{Same as } \text{Pr} \ll 1.$$



Hence:

$$\boxed{\left(\frac{X_T / D_h}{\text{Pr} \cdot \text{Re}_{D_h}} \right)^{\frac{1}{2}} \sim \frac{1}{2}}$$

\Rightarrow For all Pr.

Dividing equations ① and ②:

$$\frac{X_T}{X_{el}} \sim \text{Pr}$$

For all Pr.

Following this up to see how heat transfer varies:

$$Nu = \frac{hD_h}{k} \sim \frac{q''}{\Delta T} \cdot \frac{D_h}{k} \sim \frac{k \frac{\Delta T}{\delta_T} \cdot D_h}{\Delta T \cdot k} \sim \frac{D_h}{\delta_T}$$

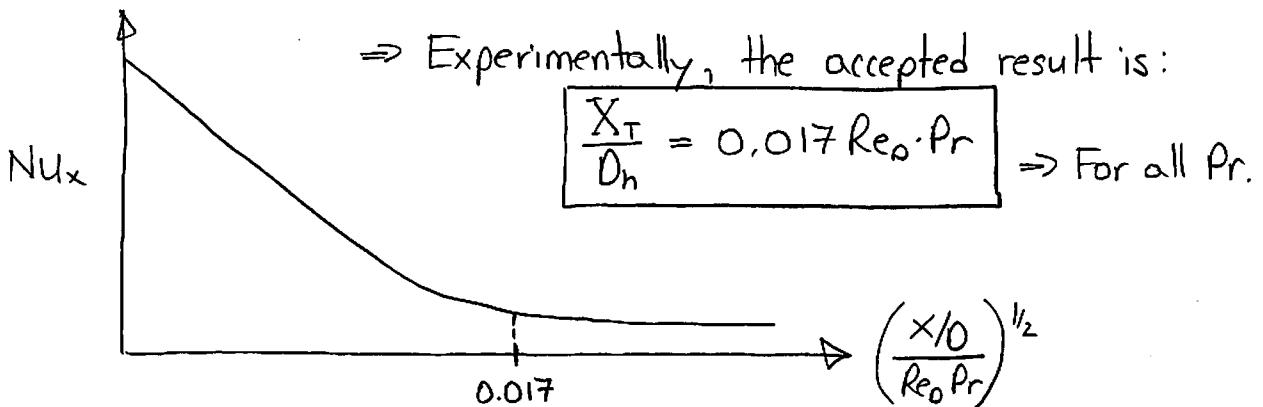
$$S_T \sim \propto \text{Pr}^{-1/2} \text{Re}_x^{-1/2} \Rightarrow \text{Convert } \text{Re}_x \text{ to } \text{Re}_{D_h}$$

$$Nu \sim \left(\frac{x/D_h}{\text{Re}_{D_h} \cdot \text{Pr}} \right)^{-1/2}$$

\Rightarrow Laminar

\Rightarrow Developing flow, for all Pr

Typically our results are plotted as:



$$\frac{X_T}{D_h} = 0.017 \text{Re}_0 \cdot \text{Pr}$$

For all Pr.

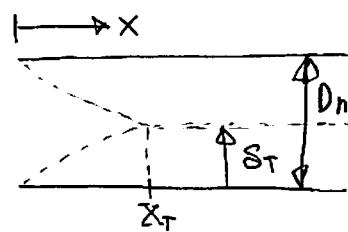
Fully Developed Region ($x > X_{el}$ and $x > X_T$)

Before we do any analysis, let's try scaling:

$$Nu = \frac{hD_h}{k} \sim \frac{D_h}{\delta_T}$$

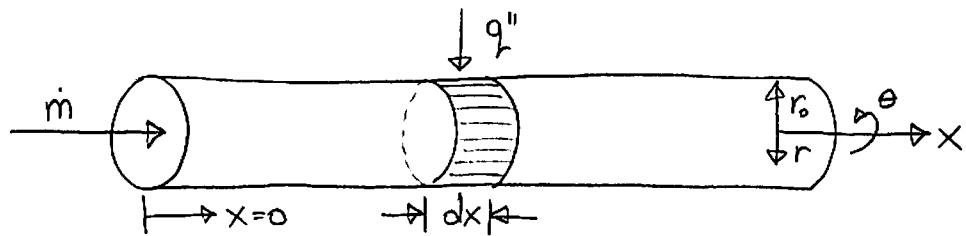
For the fully developed region, $S_T \sim \frac{D_h}{2}$

$$Nu \sim \frac{D_h}{D_h/2} \sim 2 \Rightarrow Nu \sim 2$$



We can now try to analyze the problem more exactly

Consider a pipe section of length dx undergoing heat transfer



$$\text{We know that } \dot{m} = \rho \pi r_o^2 \bar{U}$$

Applying an energy balance on our sliver (dx)

$$q'' \cdot 2\pi r_o dx = \dot{m}(h_{x+dx} - h_x); \quad h_x = \text{fluid enthalpy at } x$$

For most fluids, we can model them as an ideal gas:

$$dh = c_p dT_m \Rightarrow \text{Back substituting}$$

$$\boxed{\frac{dT_m}{dx} = \frac{2}{r_o} \frac{q''}{\rho c_p \bar{U}}} \quad ①$$

Note, T_m is defined as the mean temperature of the fluid as defined by thermodynamics, where the term "bulk" is used.

But we know that the fluid temperature varies as a function of x and r . However, we've defined our dh relation with respect to the thermodynamic bulk temperature.

$$q'' \cdot 2\pi r_o dx = \dot{m} dh = d \iint_A \rho u c_p T dA \Rightarrow \text{since both } u \text{ and } T \text{ are functions of } r.$$

Back substituting ① into the equation above:

$$\frac{\rho c_p \bar{U} r_o}{2} \cdot \frac{\partial T_m}{\partial x} \cdot 2\pi r_o dx = d \iint_A \rho u c_p T dA$$

$$\int \rho c_p \pi r_o^2 \bar{U} dT_m = \int d \iint \rho u c_p T dA$$

$$\rho c_p \bar{U} A T_m = \iint_A \rho u c_p T dA$$

For constant property fluids :

$$T_m = \frac{1}{\pi r_o^2 \bar{U}} \int_0^{2\pi} \int_0^{r_o} u T r dr d\theta$$

We can simplify this further since θ is usually not a factor

$$T_m = \frac{1}{\pi r_o^2 \bar{U}} \int_0^{r_o} u T 2\pi r dr$$

⇒ Pipe flow, constant prop.

Since the fluid temperature varies over the cross section, it is customary to define heat transfer analysis with respect to the mean temperature, T_m :

$$h = \frac{q''|_o}{T_o - T_m} = \frac{k \frac{\partial T}{\partial r}|_o}{T_o - T_m}$$

Fully Developed Temperature Profile

Let's analyze a round tube (θ -symmetric). Our energy equation becomes:

$$\frac{1}{\alpha} \left(U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial r} \right) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2}$$

For the hydrodynamic and thermally fully developed region, we know that: $V=0$ and $U=U(r)$ only:

$$\underbrace{\frac{U(r)}{\alpha} \frac{\partial T}{\partial x}}_{\text{Convection}} = \underbrace{\frac{\partial^2 T}{\partial r^2}_{\text{radial}} + \frac{1}{r} \frac{\partial T}{\partial r}}_{\text{Conduction}} + \frac{\partial^2 T}{\partial x^2}_{\text{axial}}$$

Using scaling, we can see which terms dominate and when,

We know: $U \sim \bar{U}$

$$\frac{dT_m}{dx} \sim \frac{2q''}{r_0 \rho c_p \bar{U}} ; \quad \frac{\partial^2 T}{\partial x^2} \sim \frac{\partial^2 T_m}{\partial x^2} \sim \frac{1}{x} \left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right)$$

$$\frac{\partial^2 T}{\partial r^2} \sim \frac{\Delta T}{r_0^2} ; \quad \frac{1}{r} \frac{\partial T}{\partial r} \sim \frac{1}{r_0} \frac{\Delta T}{r_0} \sim \frac{\Delta T}{r_0^2}$$

Back substituting:

$$\frac{\bar{U}}{\alpha} \left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right) \sim \frac{\Delta T}{r_0^2} + \frac{1}{x} \left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right)$$

We know the radial conduction term must be present in order for us to solve the problem (otherwise its a trivial solution). Multiplying through by $r_0^2 / \Delta T$

$$\frac{\bar{U}}{\alpha} \frac{r_0^2}{\Delta T} \left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right) \sim \frac{r_0^2}{\Delta T} \frac{\Delta T}{r_0^2} + \frac{1}{x} \frac{r_0^2}{\Delta T} \left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right)$$

$$\frac{hD}{k} \sim 1 + \underbrace{\frac{q''}{\Delta T} \left(\frac{1}{r_0 \rho c_p \bar{U}} \right)}_{h} \underbrace{\left(\frac{2q''}{r_0 \rho c_p \bar{U}} \right) \frac{r_0}{\alpha}}_{\frac{\alpha}{k}} h$$

$$\underbrace{\frac{hD}{k}}_{\text{Convection}} \sim 1 + \underbrace{\left(\frac{hD}{k} \right)^2 \left(\frac{\alpha}{\bar{U} D} \right)^2}_{\text{Conduction}}$$

Comparing our first and third scales, we see that, if

$$\frac{\bar{U} D}{\alpha} \gg 1 \Rightarrow \text{Longitudinal conduction is negligible}$$

We already know this number:

$$Pe_0 = \frac{\bar{U} D}{\alpha} = \text{Peclet \#} = \frac{\text{advection}}{\text{conduction}}$$

\hookrightarrow convection = conduction + advection

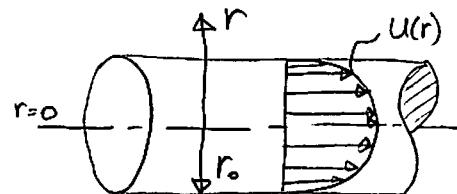
We already know from previous scaling that: $Nu = \frac{hD}{k} \sim 2$, here, we see that $Nu \sim 1$ (makes sense). The difference is in the first scaling, we used r_0 instead of $D \sim r_0$.

Assuming that $\rho c_p \gg 1$, our governing equation becomes:

$$\frac{u(r)}{\alpha} \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}$$

sometimes written as:

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$



We already solved that $u = 2\bar{u} \left(1 - \frac{r^2}{r_0^2} \right)$ for a round tube

Our B.C.'s are: $\frac{\partial T}{\partial r} \Big|_{r_0} = 0$

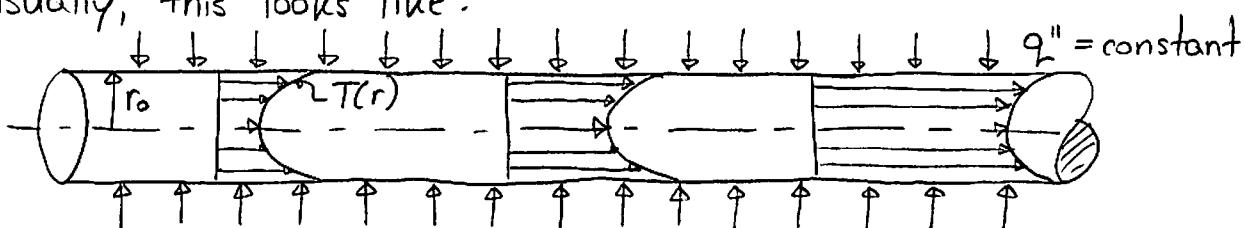
$T(r=r_0) = T_0(x) \Rightarrow \text{Constant heat flux case}$

Let: $\frac{T - T_m}{T_0 - T_m} = \phi(r) \Rightarrow q'' \Big|_{r_0} = -k \frac{\partial T}{\partial r} \Big|_{r_0} = -k \frac{\partial \phi}{\partial r} \Big|_{r_0} \cdot (T_0 - T_m) = \text{constant}$

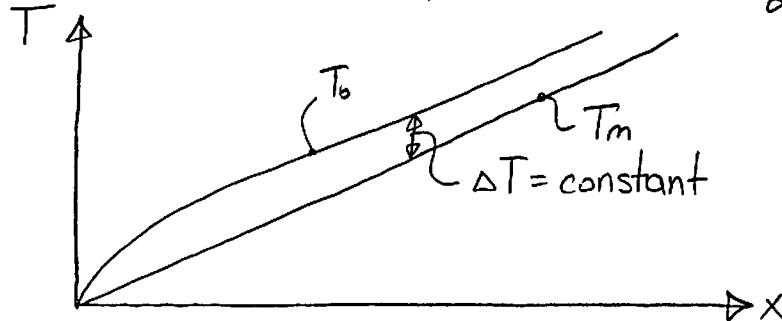
Here, we realize that the temperature profile doesn't change:

$$q'' \Big|_{r_0} = \text{constant} = -k \underbrace{\frac{\partial \phi}{\partial r} \Big|_{r_0}}_{\text{constant}} \underbrace{(T_0 - T_m)}_{\text{has to be constant since } \frac{\partial \phi}{\partial r} \Big|_{r_0} = \text{constant}} \rightarrow \frac{\partial}{\partial x} (T_0 - T_m) = \frac{\partial}{\partial x} (\text{const.}) \rightarrow \frac{\partial T_0}{\partial x} = \frac{\partial T_m}{\partial x}$$

Visually, this looks like:



This implies that for $q'' = \text{constant}$, $\frac{\partial T_m}{\partial x} = \frac{\partial T_0}{\partial x}$



\Rightarrow if $\frac{\partial T_m}{\partial x} \neq \frac{\partial T_0}{\partial x}$, then our two lines would cross and this would violate the first law of thermodynamics.

We also know from before that: $\frac{\partial T}{\partial x} = \frac{\partial T_m}{\partial x} = \frac{q''}{\rho C_p r_0 U} = \text{constant}$

Back substituting $u(r)$ into our PDE

$$2U \left[1 - \frac{r^2}{r_0^2} \right] \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Rightarrow \text{PDE since } T(x, r)$$

Back substituting $\frac{\partial T}{\partial x}$:

$$\left(1 - \frac{r^2}{r_0^2} \right) \frac{4q'' U}{\rho C_p r_0 U} = \frac{k}{\rho C_p r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Rightarrow \text{ODE} \Rightarrow \text{Solvable}$$

$$\boxed{\left(1 - \frac{r^2}{r_0^2} \right) \frac{4q''}{k r_0} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)} \quad ①$$

Integrating once, we obtain

$$r \frac{\partial T}{\partial r} = \frac{4q''}{kr_0} \left[\frac{r^2}{2} - \frac{r^4}{4r_0^2} \right] + C_1$$

$$\frac{\partial T}{\partial r} = \frac{4q''}{4r_0} \left[\frac{r}{2} - \frac{r^3}{4r_0^2} \right] + \frac{C_1}{r} \leftarrow C_1 = 0 \text{ since } \frac{C_1}{r} \rightarrow \infty \text{ as } r \rightarrow 0$$

Integrating once more:

$$T(r) = \frac{4q''}{kr_0} \left[\frac{r^2}{4} - \frac{r^4}{16r_0^2} \right] + C_2$$

Using our second B.C.: $T(r=r_0) = T_0 \Rightarrow C_2 = T_0 - \frac{4q''}{kr_0} \left[\frac{3}{16} r_0^2 \right]$
So our solution becomes:

$$\boxed{T(r) = T_0 - \frac{4q''}{kr_0} \left(\frac{3r_0^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_0^2} \right)} \quad ②$$

\Rightarrow Quartic temperature profile for $q'' = \text{const.}$

Solving for T_m :

$$T_m = \frac{\int_0^{r_0} U T 2\pi r dr}{\pi r_0^2 U} \Rightarrow \boxed{T_m = T_0 - \frac{11}{24} \cdot \frac{q'' r_0}{k}}$$

Note, $(T_m - T_0) = -\frac{11}{24} \cdot \frac{q'' r_0}{k} = \text{constant} \Rightarrow \text{makes sense according to our formulation.}$

Solving for Nusselt number:

$$h = \frac{q''}{T_0 - T_m} = \frac{24k}{\pi r_0} = \frac{48k}{\pi D}$$

$$Nu_0 = \frac{hD}{k} = \frac{48}{\pi} = 4.364$$

$Nu_0 = 4.364 \Rightarrow$ Laminar flow in tube with constant heat flux.

Note, initially we solved that $Nu_0 \sim 2$. Pretty close right!

Uniform Wall Temperature ($T_0 = \text{constant}$)

For constant wall temperature, the solution proves difficult to solve analytically. Typically numerical solutions are obtained.

$q(x) = h [T_0 - T_m] \Rightarrow T_m = f(x)$, $T_0 = \text{constant}$
we know from before that:

$$\frac{2T_m}{\partial x} = \frac{2q''}{r_0 \rho C_p U} \Rightarrow q'' = \frac{r_0 \rho C_p U}{2} \frac{\partial T_m}{\partial x} = h [T_0 - T_m]$$

let $\Theta = T_0 - T_m \Rightarrow d\Theta = -dT_m$

$$-\frac{r_0 \rho C_p U}{2} \frac{\partial \Theta}{\partial x} = h\Theta \Rightarrow \text{We can solve this}$$

$$\int_{\Theta_1}^{\Theta} \frac{\partial \Theta}{\Theta} = - \int_{x_1}^x \frac{2h}{r_0 \rho C_p U} dx$$

$$\ln \left(\frac{T_0 - T_m}{T_0 - T_{m,1}} \right) = - \frac{2h}{r_0 \rho C_p U} (x - x_1)$$

$$T_0 - T_m = (T_0 - T_{m,1}) \exp \left(- \frac{2h}{r_0 \rho C_p U} (x - x_1) \right) \Rightarrow \frac{1}{\rho C_p} = \frac{\alpha}{k}$$

$T_0 - T_m = (T_0 - T_{m,1}) \exp \left(- \frac{\alpha Nu}{r_0^2 U} (x - x_1) \right) \quad ①$

↳ The temperature difference decreases exponentially in the flow direction. As does the heat flux, q''

$\phi(r)$ only $\neq f(x)$
scaling term.

Now, we can say that: $T = T_0 - \phi(T_0 - T_m)$

$$\frac{\partial T}{\partial x} = \frac{2}{\partial x} [T_0 - \phi(T_0 - T_m)] = -\frac{\partial \phi}{\partial x} (T_0 - T_m) + \phi \frac{\partial T_m}{\partial x}$$

But since we have a fully developed flow: $\frac{\partial \phi}{\partial x} = 0$

$$\frac{\partial T}{\partial x} = \phi \frac{\partial T_m}{\partial x}$$

Back substituting everything into our energy equation:

$$-2Nu(1-r^2)\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \phi}{\partial r} \quad (2)$$

↳ Similar to our $q''=\text{constant}$ case but now ϕ term appears on the LHS.

We also need to use both boundary conditions to solve this numerically

$$\frac{\partial \phi}{\partial r} = 0 \text{ at } r=0 \quad (3)$$

$$\phi = 0 \text{ at } r=r_0 \quad (4)$$

Our last condition needed is our definition of Nu, since $\phi = f(r, Nu)$ in this problem:

$$Nu = \frac{hD}{k} = \frac{q''}{\Delta T} \cdot \frac{D}{k} = -\frac{k \frac{\partial T}{\partial r}|_{r_0}}{\Delta T} \cdot \frac{D}{k} = -\frac{\partial T}{\partial r}|_{r_0} \cdot D \cdot \frac{1}{T_0 - T_m}$$

We know $T = T_0 - \phi(T_0 - T_m)$

$$\frac{\partial T}{\partial r} = -\frac{\partial \phi}{\partial r} (T_0 - T_m) + \phi \frac{\partial}{\partial r} (T_0 - T_m) \xrightarrow{0} T_0, T_m \neq f(r)$$

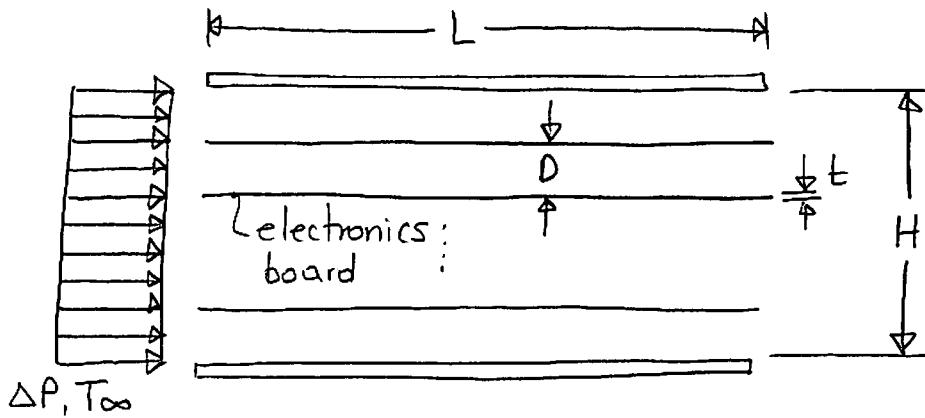
$$Nu = f\left(-\frac{\partial \phi}{\partial r}\right) (T_0 - T_m) \cdot 0 \cdot \frac{1}{(T_0 - T_m)}$$

$$Nu = 2r_0 \cdot \frac{\partial \phi}{\partial r} \quad (5)$$

To solve (2), we need (3), (4) & (5). Solving numerically, we obtain:

$$Nu_0 = \frac{hD}{k} = 3.66 \Rightarrow \text{Laminar flow in tube with constant wall temperature.} \quad (136)$$

Stack of Heat Generating Plates (Electronics cooling example)
 Think of a stack (or rack) of electronics circuit boards



Incoming cooling fluid has a constant T_∞ , and pressure drop budget from a supply fan or pump.

Assuming the flow is laminar, and board temperature $T_w = \text{constant}$ (< the electronics component breakdown temperature).
 Also: $L \ll D$.

$$n = \frac{H}{D} \quad (\text{assuming } L \ll D)$$

Small Spacing limit: $D \rightarrow 0$

For $D \rightarrow 0$, we can say each channel becomes fully developed rapidly and remains fully developed for all L . Also, the fluid outlet temperature approaches T_w .

$$\bar{U} = \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \Rightarrow \text{pg. 123 of notes} \Rightarrow \text{fully developed channel flow.}$$

$$\dot{m}' = \rho \bar{U} H = \rho H \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \quad (\dot{m}' = \text{mass flow rate per unit depth})$$

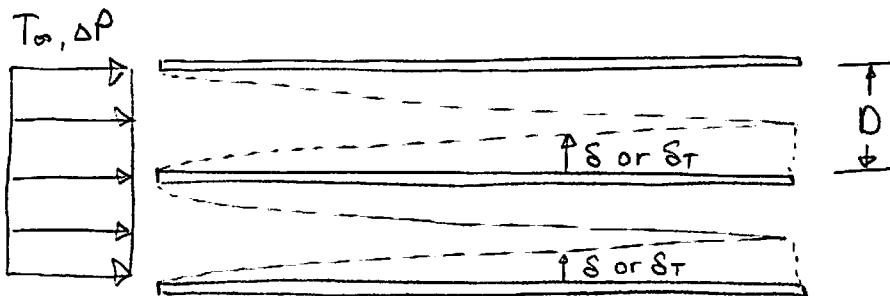
For total heat transfer, since $T = T_w$ at $x=L$

$$q'_a = \dot{m}' C_p (T_w - T_\infty) = \rho H \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \cdot C_p (T_w - T_\infty)$$

So $q'_a \sim D^2$ for the small spacing limit.

Large Spacing Limit: $D \rightarrow \infty$

For large spacing, each channel looks like the channel entrance region for the whole length:



$$x_T \text{ & } x_{eL} = L$$

Here, ΔP is fixed, so we need to solve for U_∞ that can achieve entrance effects for the whole plate.

A force balance on the whole control volume ($H \times L$) reveals:

$$\underbrace{\Delta P \cdot H}_{\text{Pressure Drop}} = \underbrace{n \cdot 2 \bar{\tau}_o \cdot L}_{\text{Total shear force}} \Rightarrow \bar{\tau}_o = \text{averaged shear stress over } L.$$

$$\bar{\tau}_o = 1.328 Re_L^{-1/2} \cdot \frac{1}{2} \rho U_\infty^2$$

Back substituting:

$$U_\infty = \left(\frac{1}{1.328} \cdot \frac{\Delta P H}{n L^{1/2} \rho U^{1/2}} \right)^{2/3}$$

For the overall heat transfer rate from one board

$$\frac{hL}{k} = \frac{q''}{T_w - T_\infty} \cdot \frac{L}{k} = 0.664 Pr^{1/3} \left(\frac{U_\infty L}{D} \right)^{1/2}; \quad Pr > 0.5$$

$$q'_1 = q'' \cdot L = k (T_w - T_\infty) 0.664 Pr^{1/3} \left(\frac{U_\infty L}{D} \right)^{1/2}$$

Assuming both sides are heating and maintained at T_w :

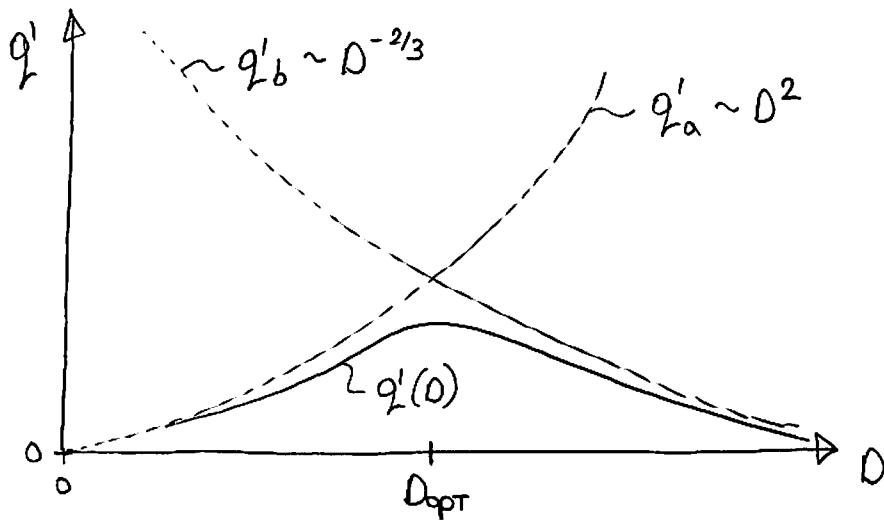
$$q'_b = 2nq'_1 = 2nk (T_w - T_\infty) 0.664 Pr^{1/3} \left(\frac{U_\infty L}{D} \right)^{1/2}$$

But we know that $n = \frac{H}{D}$, and we solved for U_∞

$$q'_b = 1.208 k (T_w - T_\infty) H \frac{\Pr^{1/3} L^{1/3} \Delta P^{1/3}}{\rho^{1/3} U^{2/3} D^{2/3}}$$

We see that $q'_b \sim D^{-2/3}$ in the large spacing limit.

This problem is a classical example of solution via intersection of asymptotes. We can do this since our two solutions are limiting cases only, and in between, mixed behaviour will occur:



To obtain our solution, we can equate the two solutions via scaling

$$q'_a \sim q'_b \Rightarrow \rho H \frac{D^2}{12U} \cdot \frac{\Delta P}{L} C_p (T_w - T_\infty) \sim 1.08 k (T_w - T_\infty) H \frac{(\Pr L \Delta P)^{1/3}}{\rho^{1/3} U^{2/3} D^{2/3}}$$

$$D_{opt} \approx 2.73 L Be_L^{-1/4} \quad \text{for } 0.7 < \Pr < 10^3$$

$$Be_L = \frac{\Delta P L^2}{U \alpha} = \text{Bejan \# (Dimensionless } \Delta P)$$

Note: $D_{opt, exp} = 3.05 L Be_L^{-1/4}$.

Results show for this solution that the board length (L) is of the same order of magnitude as the thermal entrance length (X_T).

Solving for our maximum heat transfer at D_{opt} , we obtain:

$$q'_{max} \leq 0.62 \left(\frac{\rho \Delta P}{\rho_r} \right)^{1/2} \cdot H c_p (T_w - T_\infty) \Rightarrow T_w = \text{constant}$$

Two other cases that are useful to know are $q'' = \text{constant}$

$$D_{opt} = 3.2 L Be_L^{-1/4} \Rightarrow q'' = \text{constant}$$

$$q''_{max} \leq 0.4 \left(\frac{\rho \Delta P}{\rho_r} \right)^{1/2} \cdot H \cdot c_p (T_{w,L} - T_\infty) ; T_{w,L} = T_w(L) \Rightarrow \text{max temp.}$$

For one side of each board $T_w = \text{constant}$, and the other adiabatic

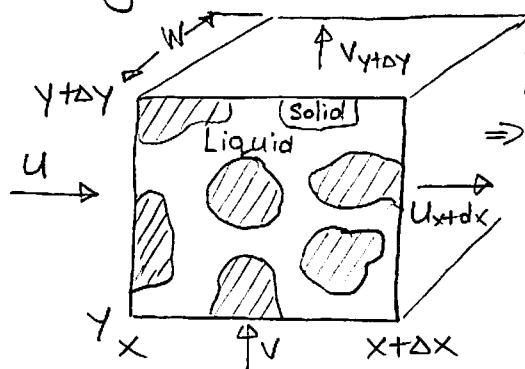
$$D_{opt} = 2.10 L Be_L^{-1/4}$$

$$q''_{max} \leq 0.37 \left(\frac{\rho \Delta P}{\rho_r} \right)^{1/2} H c_p (T_w - T_\infty) \Rightarrow T_w = \text{constant on 1 side} \\ q'' = 0 \text{ on other.}$$

Convection in Porous Media

Relatively old problem due to need to manage the water table for irrigation systems.

Assuming we have a homogeneous porous medium:



\Rightarrow We can estimate the flow to be 20 but locally it is always 30 in nature \Rightarrow just like turbulence

We assume $W \gg \Delta x$ and $W \gg \Delta y$ (2D flow). Only rates in the x & y directions are important:

$$\underbrace{\Delta x W}_{y \text{ cross-sect.}} \text{ and } \underbrace{\Delta y W}_{x \text{-cross sect.}} \Rightarrow \underbrace{\Delta x \Delta y}_{z \text{ cross section}}$$

$$\dot{m}_x = \rho \int_y^{y+\Delta y} \int_0^W u_p dz dy \Rightarrow u_p = \text{uneven } x\text{-velocity distribution over void patches in } x\text{-plane.}$$

To make our lives simpler, we can determine the area averaged x -velocity:

$$U = \frac{1}{W \Delta y} \int_0^{y+\Delta y} \int_0^W u_p(y, z) dz dy \Rightarrow \dot{m}_x = \rho U (W \Delta y) \quad (1)$$

For the y -direction, we can do the same:

$$V = \frac{1}{W \Delta x} \int_0^{x+\Delta x} \int_0^W v_p(x, z) dz dx \Rightarrow \dot{m}_y = \rho V (W \Delta x) \quad (2)$$

Note, we've assumed that ρ is constant in the $\Delta x \Delta y$ domain, not necessarily over the entire x, y domain.

Applying mass conservation:

$$\frac{\partial M_{cv}}{\partial t} = \sum_{inlet} \dot{m} - \sum_{outlet} \dot{m} \quad \text{③} \Rightarrow M_{cv} \text{ is the instantaneous mass of the C.V.}$$

We can define $M_{cv} = \rho W \Delta x \Delta y \phi \quad \text{④}$

$$\phi \equiv \text{porosity or void fraction} = \frac{\text{void volume}}{\text{total volume}}$$

Combining ③ & ④, we obtain:

$$\frac{\partial}{\partial t} (\rho \phi W \Delta x \Delta y) + \frac{\partial \dot{m}_x}{\partial x} \Delta x + \frac{\partial \dot{m}_y}{\partial y} \Delta y + \text{H.O.T.}(\Delta x^n, \Delta y^n) = 0$$

Back substituting ① and ②

$$\frac{\partial}{\partial t} (\rho \phi W \Delta x \Delta y) + \frac{\partial}{\partial x} (\rho u W \phi) \Delta x + \frac{\partial}{\partial y} (\rho v W \phi) \Delta y + \text{H.O.T.} = 0$$

Divide through by $\Delta x \Delta y$ and let $\Delta x, \Delta y \rightarrow 0$; H.O.T. $\rightarrow 0$

$$\phi \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

In general:

$$\phi \frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

\mathbf{v} = volume averaged velocity vector (u, v, w)

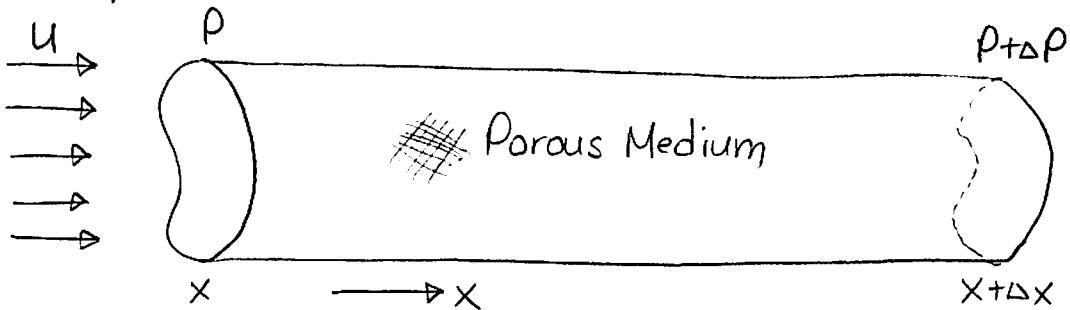
Note, if medium is a pure fluid, $\phi=1$, and we return back to our original definition of mass conservation.

Darcy's Law

The main constitutive relation for flow through porous media is Darcy's law. Darcy, a french hydrologist empirically derived it by studying the flow in sand packed beds. It is the equivalent to Fourier's law in heat transfer:

$$u = \frac{K}{\mu} \left(-\frac{\partial P}{\partial x} \right) \quad (1), \quad K = \text{permeability}$$

This law can be derived from Navier - Stokes equation, however it requires the permeability tensor and is skipped here.



Looking at the dimension of K :

$$[K] = \frac{[u][u]}{\left[-\frac{\partial P}{\partial x} \right]} = (\text{length})^2$$

Note the similarity of equation (1) & our Hagen - Poiseulle flow solution; For a pipe and channel

$$\bar{u} = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x} \right)$$

Pipe

$$\bar{u} = \frac{D^2}{12\mu} \left(-\frac{\partial P}{\partial x} \right)$$

Channel

$$K \sim r_0^2 \sim D^2$$

So: $K^{1/2} \equiv \text{length scale of the pore diameter}$.

By assuming a small scale bundle of channels with H-P flow, we can derive (1) from it.

Defining our Reynolds number based on our pore scale:

$$\boxed{Re = \frac{\rho u K^{1/2}}{u}} \quad (2)$$

And our porous flow friction factor:

$$\boxed{f = \frac{\left(-\frac{\partial P}{\partial x}\right) K^{1/2}}{\rho u^2}} \quad (3)$$

\Rightarrow Before for H-P flow, we had
 $f = \frac{\left(\frac{\Delta P}{L}\right) D^{1/2}}{\frac{1}{2} \rho u^2}$

Back substituting our definition for u (1):

$$Re = \frac{\rho \frac{K}{u} \left(-\frac{\partial P}{\partial x}\right) K^{1/2}}{u} = \frac{\rho K^{3/2} \left(-\frac{\partial P}{\partial x}\right)}{u^2} \quad (4)$$

$$f = \frac{\left(-\frac{\partial P}{\partial x}\right) K^{1/2}}{\rho \left(\frac{K^{3/2}}{u^2} \left(-\frac{\partial P}{\partial x}\right)\right)} = \frac{u^2}{\rho K^{3/2} \left(-\frac{\partial P}{\partial x}\right)} \quad (5)$$

We see that: $\boxed{f = \frac{1}{Re}}$ \Rightarrow Second form of Darcy's Law.

\hookrightarrow Valid for laminar flow $\Rightarrow Re \leq 10$

Note, for $Re > 10$, inertia becomes important and we can use the Forchheimer modification:

$$\boxed{-\frac{\partial P}{\partial x} = \frac{u}{K} u + b \rho u / u}$$

b = empirical constant based on geometry of pores.

From this stems:

$$\boxed{f = \frac{1}{Re} + 0.55} \Rightarrow Re > 10$$

If we have gravity present (body force = ρg_x)

$$u = \frac{K}{\mu} \left(-\frac{\partial P}{\partial x} + \rho g_x \right) \Rightarrow \text{Note, } u=0 \text{ when } \frac{\partial P}{\partial x} = \rho g_x \\ \text{Pressure matches hydrostatics.}$$

or in 3D:

$$\mathbf{V} = \frac{K}{\mu} (-\nabla P + \rho \mathbf{g}) ; \quad \mathbf{V} = (u, v, w) \\ \mathbf{g} = (g_x, g_y, g_z)$$

In many typical problems involving seepage flow of water in the ground, ρ and $\mu = \text{constant}$, $\mathbf{g} = (0, -g, 0)$

$$\mathbf{V} = -\frac{K}{\mu} \nabla E \Rightarrow E = P + \rho g y$$

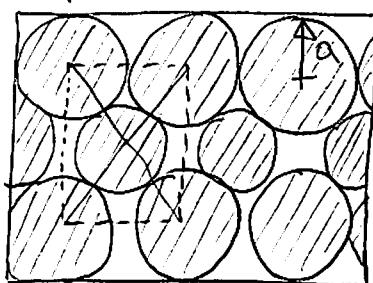
Our mass conservation for $P = \text{const.}$ becomes: $\nabla \cdot \mathbf{V} = 0$
Combining, we obtain:

$$\nabla^2 E = 0 \Leftrightarrow \nabla^2 T = 0 \Leftrightarrow \nabla^2 \phi = 0$$

Can solve the seepage problem using steady state heat conduction.

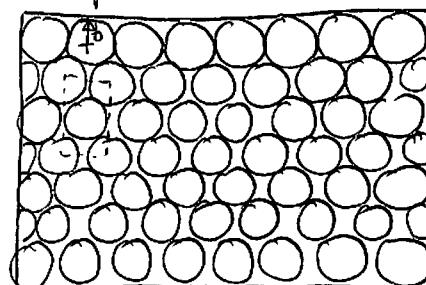
Note, one important thing to discuss is the difference between permeability and porosity.

Spheres Diameter a



or

Spheres Diameter b



$$\phi_a \equiv \text{porosity} = \frac{2(\pi a^3)}{(2a)(3.464a)}$$

$$\phi_a = 0.91$$

$$K_a \approx a$$

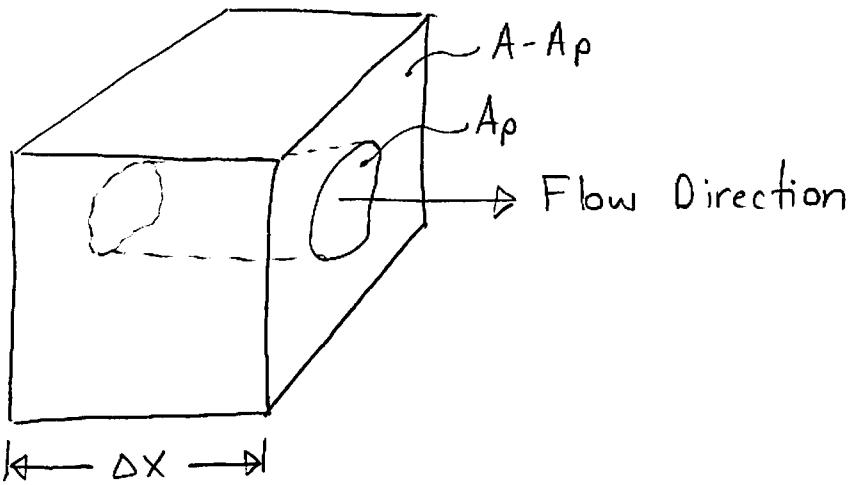
$$\phi_b \equiv \text{porosity} = \frac{2(\pi b^3)}{(2b)(3.464b)}$$

$$\phi_b = 0.91 = \phi_a$$

$$K_b \approx b$$

Heat Transfer in Porous Media

Let's consider a simple 1-D model in order to obtain our energy equation in porous media:



According to our previous definition: $\phi = \frac{A_p \Delta x}{A \Delta x}$ ①

For the solid domain:

heat generation/volume

$$\rho_s C_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2} + q_s''' \Rightarrow \rho_s, C_s, k_s \Rightarrow \text{solid properties}$$

If $T = \text{constant}$ over the solid volume, we can integrate volumetrically

$$\Delta x (A - A_p) \rho_s C_s \frac{\partial T}{\partial t} = \Delta x (A - A_s) k_s \frac{\partial^2 T}{\partial x^2} + \Delta x (A - A_p) q_s''' \quad ②$$

For the fluid domain:

$$\rho_f C_{A,f} \left(\frac{\partial T}{\partial t} + u_p \frac{\partial T}{\partial x} \right) = k_f \frac{\partial^2 T}{\partial x^2} + u \underline{\phi} \quad ③$$

Note we neglected our compressibility term ($\beta T \frac{\partial P}{\partial T}$) \Rightarrow incompressible

Note, T here is the same in the fluid as the solid. Local thermal equilibrium. Not always true though so be carifull when using it.

For example \Rightarrow nuclear reactors or electronics cooling: $T_f \neq T_s$.
 If no Local Equilibrium, we must solve the fluid & solid coupled equations.
 Integrating eq. ③ over the pore volume: Note: $Au = \iint_{A_p} u dA_p$

$$\Delta x A_p \rho_f c_{pf} \frac{\partial T}{\partial t} + \Delta x A_p \rho_f c_{pf} u \frac{\partial T}{\partial x} = \Delta x A_p k_f \frac{\partial^2 T}{\partial x^2} + \Delta x u \iint_{A_p} \overline{J} dA_p \quad ④$$

Our last term represents the internal heating due to viscous dissipation.

Since this is a loss term ($E_{out} = E_{in}$), then the last term is equal to the mechanical power required to drive the flow.

We can use fluid mechanics to show that:

$$\underbrace{\Delta x u \iint_{A_p} \overline{J} dA_p}_{\text{Viscous heating}} = \underbrace{Au \left(-\frac{\partial P}{\partial x} + \rho_f g_x \right) \Delta x}_{\text{Mechanical work input}} \quad ⑤$$

Back substituting ⑤ into ④ and adding ② \Rightarrow divide out by $A \Delta x$

$$\boxed{\left[\phi \rho_f c_{pf} + (1-\phi) \rho_s c_s \right] \frac{\partial T}{\partial t} + \rho_f c_{pf} u \frac{\partial T}{\partial x}} \\ = \boxed{\left[\phi k_f + (1-\phi) k_s \right] \frac{\partial^2 T}{\partial x^2} + (1-\phi) q''_s + u \left(-\frac{\partial P}{\partial x} + \rho_f g_x \right)}$$

\hookrightarrow Porous media energy equation (incompressible)

$$\boxed{k = \phi k_f + (1-\phi) k_s} \Rightarrow \text{porous medium thermal conduct.}$$

Note, since we added ② & ④, this assumes a parallel model, hence k is valid for this assumption. In general, k needs to be experimentally measured.

We can define a capacity ratio of our medium as:

$$\sigma = \frac{\phi p_f c_{pf} + (1-\phi) p_s c_s}{p_f c_{pf}}$$

⇒ capacity ratio

Doesn't mean much, more for notation as we will see.

With this new notation, we can write: Note: $q''' = (1-\phi) q_s'''$

$$p_f c_{pf} \left(\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = k \frac{\partial^2 T}{\partial x^2} + q''' + \frac{u}{K} u^2$$

In general for 3D:

$$p_f c_{pf} \left(\sigma \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + q''' + \frac{u}{K} (\mathbf{v})^2$$

↳ $\mathbf{v} = (u, v, w)$ = volume averaged velocity vector

If heat generation & viscous heating are negligible:

$$\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

Where:

$$\alpha = \frac{k}{p_f c_{pf}}$$

= homogeneous porous medium thermal diff.

Note: Our main assumptions here are:

- 1) Homogeneous medium: solid material & fluid is distributed evenly in the medium.
- 2) Isotropic medium: i.e. $K \neq f(\text{direction})$. Note, if not the case, then:

$$u = \frac{K_x}{\alpha} \left(-\frac{\partial P}{\partial x} + \rho g_x \right), \quad v = \frac{K_y}{\alpha} \left(-\frac{\partial P}{\partial y} + \rho g_y \right), \quad w = \frac{K_z}{\alpha} \left(-\frac{\partial P}{\partial z} + \rho g_z \right)$$

$$\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_x \frac{\partial^2 T}{\partial x^2} + \alpha_y \frac{\partial^2 T}{\partial y^2} + \alpha_z \frac{\partial^2 T}{\partial z^2}$$

- 3) At any point in the medium, the solid & fluid are in local thermal equilibrium.
- 4) Local $Re = \frac{\rho u k''}{\mu} < 10$ (Darcy's Law is Applicable)
Laminar flow.

Non-LTE Heat Transfer

If the fluid and solid are not in thermal equilibrium, we must solve a coupled set of equations:

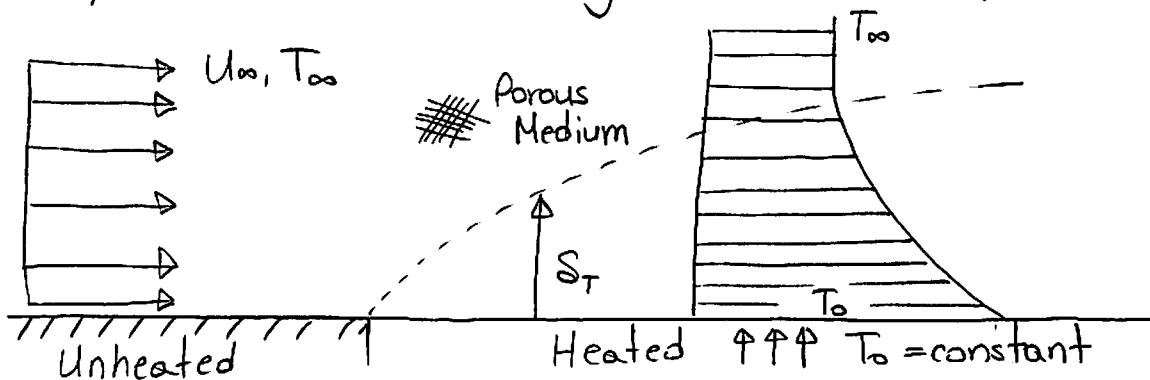
$$(1-\phi) \rho_s C_s \frac{\partial T_s}{\partial t} = (1-\phi) k_s \frac{\partial^2 T_s}{\partial x^2} + h (T_f - T_s) \quad (1)$$

$$\phi \rho_f C_f \frac{\partial T_f}{\partial t} + \rho_f C_f U \frac{\partial T}{\partial x} = \phi k_f \frac{\partial^2 T_f}{\partial x^2} - h (T_f - T_s) \quad (2)$$

Need to solve ① and ② simultaneously. Note, this example still assumes incompressible, viscous heating is negligible, and zero volumetric heating.

Boundary Layers

Let's say we had the following situation (incompressible flow)



Our governing equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{mass conservation})$$

$$u = -\frac{K}{\mu} \frac{\partial p}{\partial x} \quad ; \quad v = -\frac{K}{\mu} \frac{\partial p}{\partial y}$$

Considering that the b.l. is slender, with uniform parallel flow:

$$U = U_\infty, \quad V = 0, \quad P(x) = -\frac{U}{K} U_\infty x + \text{const}$$

↳ solved by integrating Darcy's eqn.

Applying a scaling analysis: $U \frac{\partial T}{\partial x} = \propto \frac{\partial^2 T}{\partial y^2}$ (energy eqn.)

$$\begin{aligned} U &\sim U_\infty & x &\sim x \\ \partial T &\sim \Delta T & y &\sim \delta_T \\ \partial^2 T &\sim \Delta T \end{aligned}$$

$$U_\infty \frac{\Delta T}{x} \sim \propto \frac{\Delta T}{\delta_T^2} \Rightarrow \boxed{\frac{\delta_T}{x} \sim Pe^{-1/2}}$$

$$\boxed{Nu_x = h \frac{x}{K} \sim \frac{x}{\delta_T} \sim Pe^{1/2}} \quad \boxed{Pe = \frac{U_\infty x}{\propto}} = \text{Peclet number}$$

Note, \propto = thermal diffusivity of the porous medium = $\frac{k}{\rho_f C_p}$

Note also the similarity to the Blasius - Pohlhausen solution for a flat plate. Very similar however no hydrodynamic boundary layer development due to the porous media.

Same as $Pr \ll 1$ case where $U \sim U_\infty$ (see pg. (42) of notes). Only difference is before, we used fluid properties, & here we use porous media properties.

Similarity Solution ($T_0 = \text{constant} = \text{wall temperature}$)

Assuming the following similarity variable:

$$\eta = \frac{y}{x} Pe_x^{1/2}, \quad \Theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

Converting our energy equation

$$\left. \begin{array}{l} 2T = (T_\infty - T_0) 2\theta \\ 2^2 T = (T_\infty - T_0) 2^2 \theta \\ U = U_\infty \end{array} \right\} \text{Back substitute into energy equation}$$

$$U \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \Leftrightarrow U_\infty \frac{\partial \theta}{\partial x} (T_\infty - T_0) = \alpha \frac{\partial^2 \theta}{\partial y^2} (T_\infty - T_0)$$

Now we need to convert from $x \rightarrow \eta$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial \eta}{\partial x} = \frac{2}{\alpha x} \left(\frac{y}{x} Re_x^{1/2} \right) = -\frac{y}{2} \left(\frac{U_\infty}{\alpha x^3} \right)^{1/2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{2} \left(\frac{U_\infty}{\alpha x^3} \right)^{1/2} \theta' \quad \textcircled{1}$$

Now let's deal with the right hand side:

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{2}{\alpha y} \left(\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) = \frac{2}{\alpha y} \left(\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right) \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 \theta}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2$$

$$\frac{\partial \eta}{\partial y} = \left(\frac{U_\infty}{\alpha x} \right)^{1/2} \Rightarrow \left(\frac{\partial \eta}{\partial y} \right)^2 = \frac{U_\infty}{\alpha x}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{U_\infty}{\alpha x} \theta'' \quad \textcircled{2}$$

Putting $\textcircled{1}$ & $\textcircled{2}$ together:

$$-\frac{y}{2} \left(\frac{U_\infty}{\alpha x^3} \right)^{1/2} \theta' = \frac{U_\infty}{\alpha x} \theta''$$

$$-\frac{1}{2} \frac{y}{x} \left(\frac{U_\infty x}{\alpha} \right)^{1/2} \theta' = \theta''$$

$$\boxed{\theta'' + \frac{1}{2} \eta \theta' = 0}$$

\Rightarrow ODE \Rightarrow Solvable by separation of variables

$$\begin{aligned} \text{B.C.'s} \Rightarrow \theta(0) &= 0 \\ \theta(\eta \rightarrow \infty) &= 1 \end{aligned}$$

\downarrow error function. See tables.

Solving, we obtain:

$$\boxed{\theta(\eta) = \operatorname{erf}\left(\frac{\eta}{2}\right)}$$

So for heat transfer: $q'' = -k \frac{\partial T}{\partial y} \Big|_{y=0}$

$$q'' = -k(T_\infty - T_0) \frac{\partial \theta}{\partial y} \Big|_{y=0} = -k(T_\infty - T_0) \frac{\partial \theta}{\partial z} \Big|_{z=0} \cdot \frac{\partial z}{\partial y} \Big|_{y=0}$$

$$\frac{\partial z}{\partial y} = \left(\frac{U_\infty}{\alpha x}\right)^{1/2}; \quad \frac{\partial \theta}{\partial z} \Big|_{z=0} = \pi^{-1/2} \quad (\text{from tables or Wolfram})$$

$$\boxed{q'' = -k(T_\infty - T_0)\pi^{-1/2} \cdot \left(\frac{U_\infty}{\alpha x}\right)^{1/2}} \Rightarrow \begin{aligned} &\text{Wall heat flux} \\ &\Rightarrow T_0 = \text{constant}, Re_k < 10 \end{aligned}$$

For a wall heating a fluid, $T_0 > T_\infty$

$$Nu_x = \frac{q''}{T_0 - T_\infty} \cdot \frac{x}{k} = \frac{k\pi^{-1/2} \left(\frac{U_\infty}{\alpha x}\right)^{1/2} \cdot x}{k} = \left(\frac{U_\infty x}{\pi \alpha}\right)^{1/2}$$

$$\boxed{Nu_x = 0.564 Pe_x^{1/2}} \Rightarrow T_0 = \text{constant}, Re_k < 10, \text{ Incompressible}$$

Note, in scaling analysis we had $Nu_x \sim Pe^{1/2}$ (Good!)

$$\boxed{\overline{Nu} = \frac{\overline{h}L}{k} = 1.128 Pe_L^{1/2}}$$

Constant Heat Flux

For the constant heat flux case, we can do another similarity analysis, but this time with:

$$\left. \begin{array}{l} \frac{\partial T}{\partial y} = -\frac{q''}{k} \text{ at } y=0 \\ T \rightarrow T_\infty \text{ at } y \rightarrow \infty \end{array} \right\} B.C.'s$$

$$\xi = y \left(\frac{U_\infty}{\alpha x}\right)^{1/2}; \quad \tau(\xi) = \frac{T(x,y) - T_\infty}{(q''/k)(\alpha x/U_\infty)^{1/2}}$$

Doing the math & converting our energy PDE into an ODE we obtain:

$$\left. \begin{array}{l} \tau'' + \frac{1}{2}(\xi \tau' - \tau) = 0 \\ \tau'(0) = -1 \\ \tau(\xi \rightarrow \infty) = 0 \end{array} \right\} B.C.'s$$

Differentiating once, we can do separation of variables

$$\frac{\partial}{\partial z} \left(C'' + \frac{1}{2} (3C' - C) \right) = 0$$

$$C''' + \frac{1}{2} \cancel{C'} + \frac{1}{2} 3C'' - \frac{1}{2} \cancel{C'} = 0$$

$$C''' + \frac{1}{2} 3C'' = 0 \rightarrow \frac{\partial}{\partial z} (\ln(C'')) = \frac{1}{C''} \cdot \frac{\partial}{\partial z} (C'') = \frac{C'''}{C''}$$

$$\boxed{\frac{C'''}{C''} = -\frac{1}{2} 3} \Rightarrow ODE \Rightarrow \text{Similar to Blasius: } \frac{f'''}{f''} = -\frac{1}{2} f$$

To solve, we integrate 3 times in a row and then apply our B.C.'s: (For a similar integral, see pg. 49a of notes)

$$\boxed{C(3) = \frac{2}{\pi^{1/2}} \exp\left(-\frac{3^2}{4}\right) - 3 \operatorname{erfc}\left(\frac{3}{2}\right)} \Rightarrow q'' = \text{const} \rightarrow Re_k < 10$$

Solving for our Nusselt number (I skipped a few steps here for the sake of brevity):

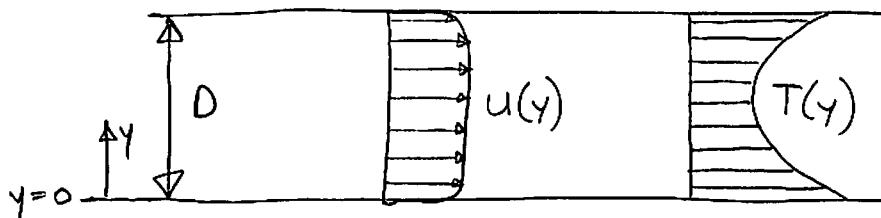
$$\boxed{Nu_x = \frac{q'' x}{k [T_0(x) - T_\infty]} = 0.866 Pe_x^{1/2}} \Rightarrow \begin{matrix} \text{Incompressible} \\ q'' = \text{constant} \\ Re_k < 10 \end{matrix}$$

$$\boxed{\overline{Nu} = \frac{q'' L}{k [\bar{T}_0 - T_\infty]} = 1.329 Pe_L^{1/2}} \Rightarrow \begin{matrix} \text{Averaged Nusselt \#} \\ \text{Remember, } k, \alpha \text{ are} \\ \text{porous media properties} \end{matrix}$$

Internal Flows filled with Porous Media

For the case of fully developed flow of a fluid through a porous medium that fills a long channel, we can use our Darcy flow model \Rightarrow volume averaged velocity u is uniform across the whole channel.

This indicates what we call "slug flow"



⇒ Different from previous analysis where $u(y) \neq \text{constant}$
see pg. (127) of notes

Note, our bulk or mean temperature reduces to :

$$\begin{aligned} T_m &= \frac{1}{UA} \cdot \int_A uT dA \Rightarrow \text{since } u(y) = \bar{U} = \text{constant} \neq f(y) \\ &= \frac{\bar{U}}{UA} \int_A T dA \end{aligned}$$

$T_m = \frac{1}{A} \int_A T dA$

⇒ Mean temperature for slug flow.
⇒ A = channel cross section.

To solve, we can use our previous approach of solving the energy equation for channel or pipe flows with $U = U_\infty = \text{constant} = \frac{k}{\mu} \left(-\frac{\partial P}{\partial x} \right)$

I've skipped the derivations but good to do for homework to check your understanding:

For Tubes (Internal Diameter = D)

$Nu_0 = \frac{q''(x)}{T_0 - T_m(x)} \cdot \frac{D}{k} = 5.78$

(tube, $T_0 = \text{constant}$)

$Nu_0 = \frac{q''}{T_0(x) - T_m(x)} \cdot \frac{D}{k} = 8$

(tube, $q'' = \text{constant}$)

For channels (Parallel Plates, spacing = D)

$Nu_0 = \frac{q''(x)}{T_0 - T_m(x)} \cdot \frac{D}{k} = 4.93$

(parallel plates, $T_0 = \text{constant}$)

$Nu_0 = 6$

(parallel plates, $q'' = \text{constant}$)

Turbulence

Looking at our governing eqn's for flow over a flat plate:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ①$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + u \underline{\Phi} \quad ②$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + v \underline{\Phi} \quad ③$$

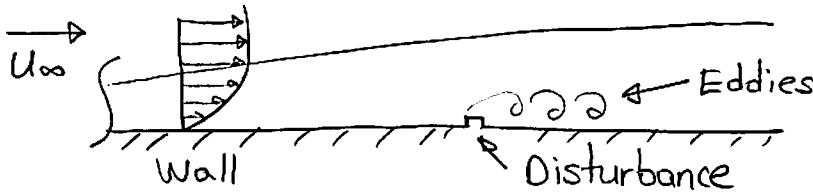
Nothing in Eqn's ①-③ suggests that a breakdown in the solution will occur at some Re_c . In fact, ①-③ are solvable for $0 < Re < \infty$.

We also know that the transition is reversible:

$$\boxed{\text{Laminar}} \rightleftharpoons \boxed{\text{Turbulent}}$$

Turbulence can be understood as a spectrum of vortices (eddies) which form to dissipate kinetic energy from large to small, until the smallest eddies dissipate via viscous shear stress.

A key to transition is small disturbances in the flow.



Note, these "disturbances" need not be physical. They may be fluctuations in the free stream velocity (U_∞), pressure gradient ($\partial P / \partial x$), or very small scale surface roughness.

Viscous Forces act to dampen the disturbances and keep laminar. Inertial Forces associated with velocity changes do the opposite.

Hence, Reynolds # is typically used as a measure of when transition will happen. \rightarrow on the order of.

$$Re = \frac{\text{Inertial Forces}}{\text{Viscous Forces}} \sim O(10^2) \text{ at transition.}$$

Viscous stability says that if we perturb the flow, the perturbation can be either damped out by the existing flow characteristics through viscous dissipation. If the existing flow does not have the ability to dampen the perturbation, transition to turbulence will occur, which has a 3D nature & more effective means to dissipate disturbances.

Laminar external flows: $Re_{cr,L} = \frac{\rho UL_{cr}}{\mu} = 3 \times 10^5 - 5 \times 10^5$

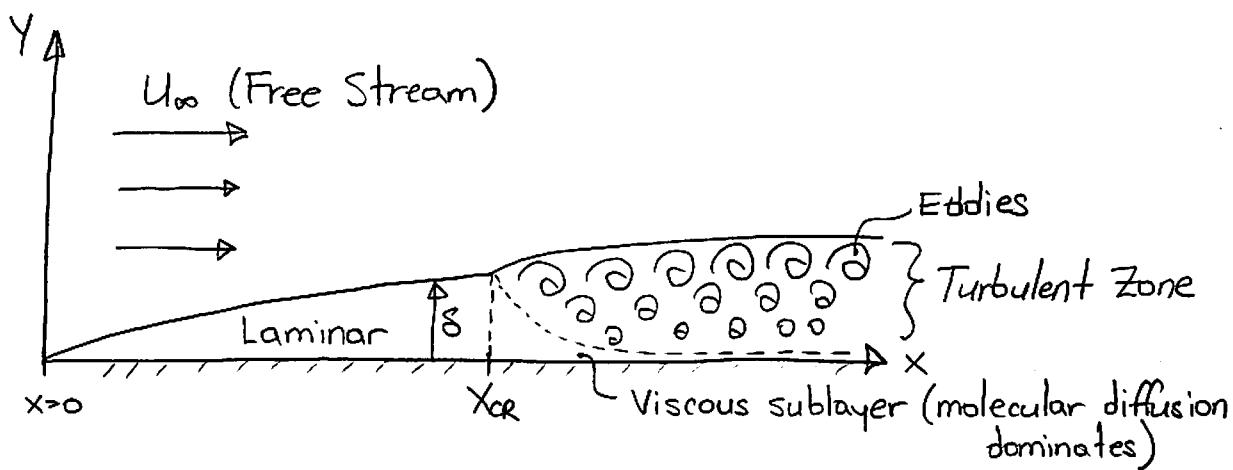
If we have large disturbances: $Re_{cr,L} = 6 \times 10^4$

Internal Flows: $Re_{cr,Dh} = \frac{\rho \overline{U} D_h}{\mu} = 2300$

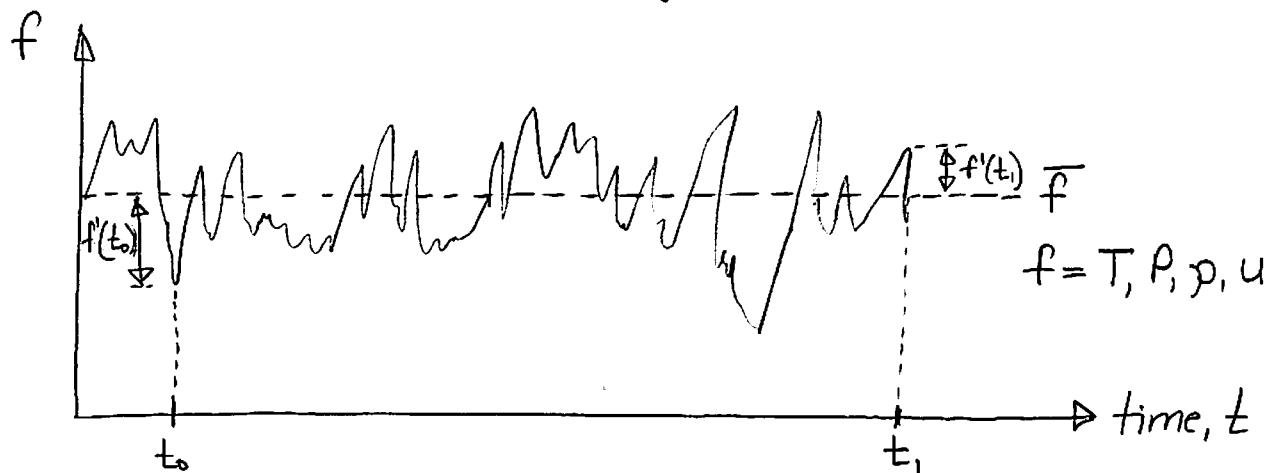
If we have no disturbances, smooth pipe; $Re_{cr,Dh} \approx 1 \times 10^5$

In general for external or internal flows, the turbulent boundary layer consists of 2 regions:

- 1) Viscous sublayer adjacent to the wall, where viscous shear & molecular diffusion govern transport
- 2) Fully turbulent zone (most of the b.l.) where velocity is not independent of time, eddies exist, and momentum & heat transfer normal to the flow is much greater than possible with viscous shear & molecular diff.



Let's deal with the fluctuating turbulent zone first:



$$f = \bar{f} + f' \Rightarrow \bar{f} \equiv \text{mean component (time averaged)} \\ f' \equiv \text{fluctuating component (time dependent)}$$

For the flows we will study:

$$U = \bar{U} + U'$$

$$V = \bar{V} + V'$$

$$W = \bar{W} + W'$$

$$\rho = \bar{\rho} + \rho'$$

$$T = \bar{T} + T'$$

ρ = constant (incompressible).

$$\bar{U} = \frac{1}{\text{period}} \int_0^{\text{period}} u dt = \frac{1}{C} \int_{t_0 - \frac{C}{2}}^{t_0 + \frac{C}{2}} u dt$$

Note also, $\int_0^{\text{period}} u' dt = 0 \Rightarrow$ fluctuating components average to zero over time.

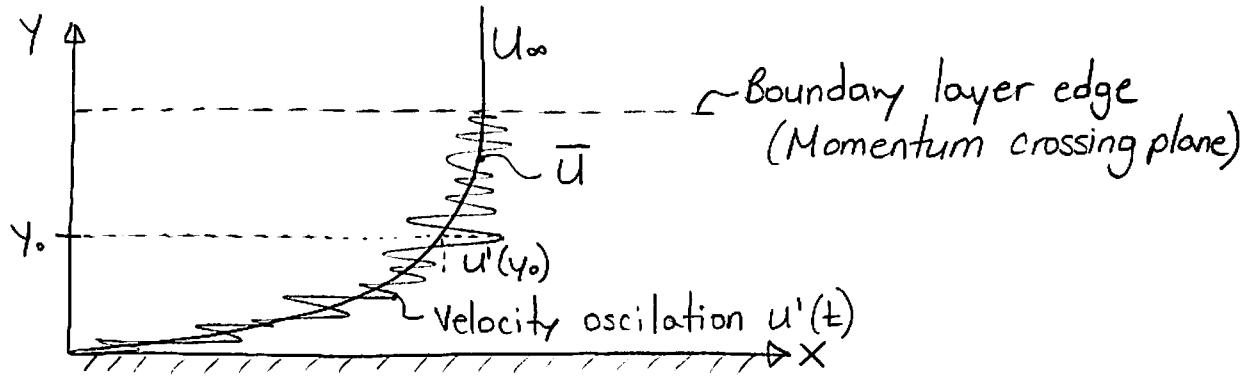
We can show this with the additive law of expectation:

$$\bar{f} = \overline{\bar{f} + f'} = \bar{f} + \overline{f'} = \bar{f} + \bar{f'} \Rightarrow \bar{f'} = 0$$

↳ since $\bar{f} = \text{constant}$ ($\bar{f} = \bar{f}$)

Also: $\overline{\bar{f} \cdot f'} = \bar{f} \cdot \overline{f'} = 0$

Let's look at our velocity boundary layer: $u = \bar{u} + u'$



The average shear stress at any y -location in the turbulent region is:

$$\overline{\tau_{xy}} = \overline{\rho(uv)}|_y \cdot \Delta x \quad (\text{momentum crossing the plane})$$

$$\overline{\tau_{xy}} = \overline{u \frac{\partial u}{\partial y}} \cdot \Delta x$$

$$\overline{\tau_{xy}} = \underbrace{\overline{u \frac{\partial u}{\partial y}}}_{\text{Laminar Component}} - \underbrace{\overline{\rho uv}}_{\text{Turbulent Component}} \Rightarrow \text{Note, we always wrote before that } \tau_{xy} = u \frac{\partial u}{\partial y}|_0 \Rightarrow \text{correct since}$$

$v=0$ at $y=0$, so $\overline{\rho uv}$ component drops. Also, $v \ll u$ in b.l. flows.

If the equation above is still confusing, think of conservation of momentum and newton's second law. Passengers hopping on two trains.

Since $\rho, \mu = \text{constant}$ (incompressible, constant property)

$$\bar{C} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u} \bar{v}$$

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

For external boundary layer flow, we can say:
empirically shown for turbulent flows

$$\bar{v} \ll \bar{u}, \text{ and } \bar{v} \ll v', \text{ and } v' = 0$$

$$\begin{aligned} \bar{u}v &= (\bar{u} + u')(\bar{v} + v') = \bar{u}\bar{v} + \bar{u}v' + u'\bar{v} + u'v' \\ &= \bar{u}\bar{v} + \underbrace{\bar{u}v'}_{0(v'=0)} + \underbrace{u'\bar{v}}_{0(\bar{u}=0)} + u'v' \end{aligned}$$

$$\bar{u}v = \bar{u}\bar{v} + u'v' \Rightarrow \text{Note: } \bar{u}' = 0 \text{ and } \bar{v}' = 0 \text{ but } u'v' \neq 0$$

since $u' \& v'$ are correlated.
Will show this below.

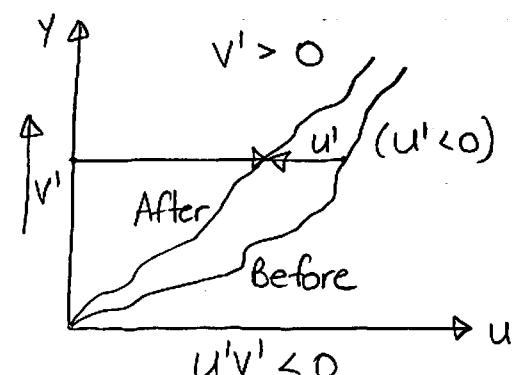
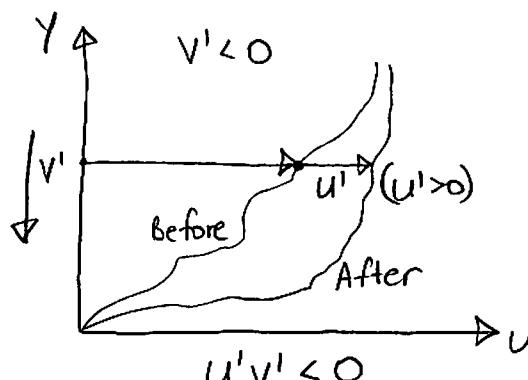
Let's see which term dominates:

$$\frac{\bar{u}\bar{v}}{u'v'} \Rightarrow \bar{v} \ll v' \text{ and } \bar{v} \ll \bar{u} \quad \left. \begin{array}{l} \bar{u} \sim v' \sim u' \end{array} \right\} \text{ Hence } \frac{\bar{u}\bar{v}}{u'v'} \ll 1$$

So we can rewrite our equation as:

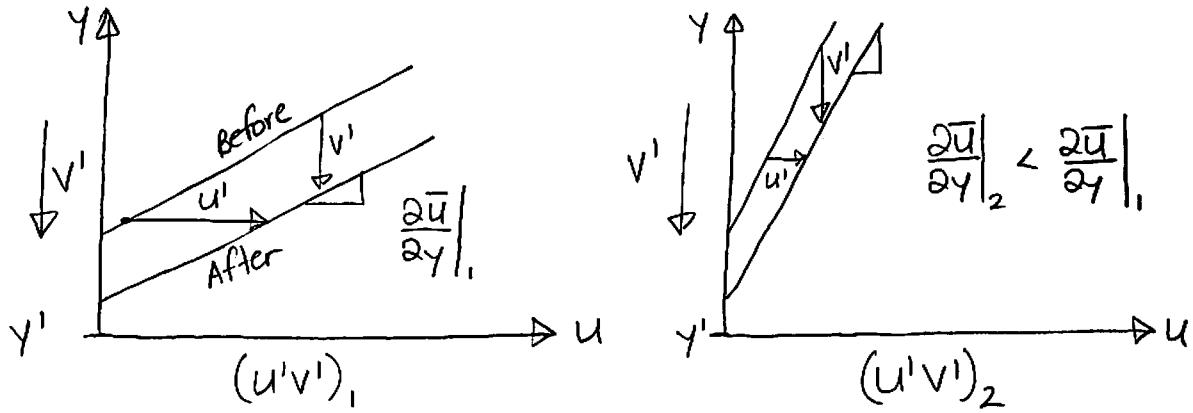
$$\boxed{\bar{C} = \mu \frac{\partial \bar{u}}{\partial y} + \rho u'v'}$$

But can we say something about the product $u'v'$?
Looking at 2 possibilities:



We see from our two cases that $u'v' < 0$ so $\bar{u}'\bar{v}' < 0$ always.

Also, since $|\bar{u}'\bar{v}'| \propto$ steepness of the profile, i.e.



We can see from above that $|(\bar{u}'\bar{v}')_1| > |(\bar{u}'\bar{v}')_2|$

Hence, we can now say: $\bar{u}'\bar{v}' \sim \frac{\partial \bar{u}}{\partial y}$

$$\bar{C} = u \frac{\partial \bar{u}}{\partial y} - \rho \bar{u}'\bar{v}'$$

$$= u \frac{\partial \bar{u}}{\partial y} - \underbrace{\left(\text{factor reflecting turbulent mixing} \right)}_{= -\rho \epsilon} \cdot \frac{\partial \bar{u}}{\partial y}$$

$$= -\rho \epsilon$$

\rightarrow negative since we know $\bar{u}'\bar{v}' < 0$,
so $\epsilon > 0$.

$$\boxed{\bar{C} = \rho (u + \epsilon) \frac{\partial \bar{u}}{\partial y}}$$

$\Rightarrow \epsilon = \text{eddy diffusivity } [\text{m}^2/\text{s}]$

We can also rewrite this as:

$$\bar{C} = (\underbrace{\rho u}_{u} + \underbrace{\rho \epsilon}_{\mu_t}) \frac{\partial \bar{u}}{\partial y}$$

u $\mu_t = \text{turbulent viscosity}$

$\bar{C} = C_{app} = \text{apparent shear stress.}$

We can also say:

$$\tau_R = \rho \bar{u}' v' = \rho \epsilon \frac{\partial \bar{u}}{\partial y} = \text{turbulent shear stress or Reynolds stress}$$

$$u' \text{ and } v' \sim \sqrt{\frac{\tau_R}{\rho}}$$

We can also say that the total shear stress τ_{app} at the wall ($\tau_R = 0$) is τ_0 .

$$u_* = \left(\frac{\tau_0}{\rho} \right)^{1/2} = \text{friction velocity} \quad \Rightarrow \text{will become important later.}$$

So now the problem is to find ϵ or u_* . We have no idea about where to start.

Before we do that, we can look back at our initial equation we started with & see more clearly how we got it:

$$\bar{\tau} = u \frac{\partial \bar{u}}{\partial y} + \rho \bar{u} \bar{v} \Rightarrow \text{Can show this more rigorously}$$

Start with mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') = 0$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0 \quad \textcircled{1}$$

Integrating eq. \textcircled{1} wrt time and noting: $\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x}$

$$\frac{1}{\text{period}} \int_0^{\text{period}} \frac{\partial \bar{u}}{\partial x} dt = \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} \Rightarrow \text{same for } \frac{\partial \bar{v}}{\partial y}$$

$$\frac{1}{\text{period}} \int_0^{\text{period}} \frac{\partial u'}{\partial x} dt = \frac{\partial u'}{\partial x} = \frac{\partial u'}{\partial x} = \frac{\partial}{\partial x} (\bar{u}') = 0$$

$$\boxed{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0} \Rightarrow \text{Turbulent conservation of mass (20)}$$

Now let's consider x-momentum:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \nabla^2 u \quad (2)$$

Note, this is just another way of writing it since:

$$\left. \begin{aligned} \frac{\partial}{\partial x}(u^2) &= 2u \frac{\partial u}{\partial x} \\ \frac{\partial}{\partial y}(uv) &= \frac{\partial u}{\partial y}v + u \frac{\partial v}{\partial y} \end{aligned} \right\} \begin{aligned} 2u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + u \frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x} \\ 2u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - u \frac{\partial u}{\partial y} &= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \end{aligned}$$

Averaging each term in (2) over time and knowing that:

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x}; \quad \frac{\partial \bar{u}}{\partial t} = 0; \quad \frac{\partial \bar{u}}{\partial t} = 0$$

$$\frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\bar{u}v) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + v \nabla^2 \bar{u} \quad (3)$$

Now we know from before that:

$$\begin{aligned} \bar{uv} &= \bar{u}\bar{v} + \bar{u}'\bar{v}' \\ \bar{u}^2 &= \bar{u}^2 + \bar{u}'^2 \end{aligned} \Rightarrow \text{showed this a few pages ago}$$

Apply to eqn. (3):

$$\frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\bar{u}v) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + v \nabla^2 \bar{u} - \frac{\partial}{\partial x}(\bar{u}^2) - \frac{\partial}{\partial y}(\bar{u}'v')$$

$$\text{We know that } \bar{u}\bar{u} = \bar{u} \cdot \bar{u} = \bar{u}\bar{u}$$

$$\cancel{\frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{u} \cancel{\frac{\partial \bar{v}}{\partial y}}} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + v \nabla^2 \bar{u} - \frac{\partial}{\partial x}(\bar{u}^2) - \frac{\partial}{\partial y}(\bar{u}'v') - \frac{\partial \bar{u}}{\partial x} \text{ (from continuity (1))}$$

So the time averaged x-momentum equation becomes: (20)

$$\boxed{\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + v \nabla^2 \bar{u} - \frac{\partial}{\partial x}(\bar{u}^2) - \frac{\partial}{\partial y}(\bar{u}'v')} \quad (4)$$

So for a boundary layer, we can simplify eq. (4)
We know that $\bar{u}' \sim v'$, and $x \sim L$, $y \sim \delta$

Scaling our two added terms:

$$\frac{\frac{\partial}{\partial x}(\bar{U}^{12})}{\frac{\partial}{\partial y}(\bar{U}'\bar{V}')} \sim \frac{\frac{\partial \bar{U}^{12}}{\partial x}}{\frac{\partial \bar{U}'\bar{V}'}{\partial y}} \sim \frac{\bar{U}^{12}}{\bar{U}'\bar{V}'} \sim \frac{s}{L} \ll 1 \Rightarrow \text{hence } \frac{\partial}{\partial x}(\bar{U}^{12}) \ll \frac{\partial}{\partial y}(\bar{U}'\bar{V}')$$

So our b.l. equation becomes:

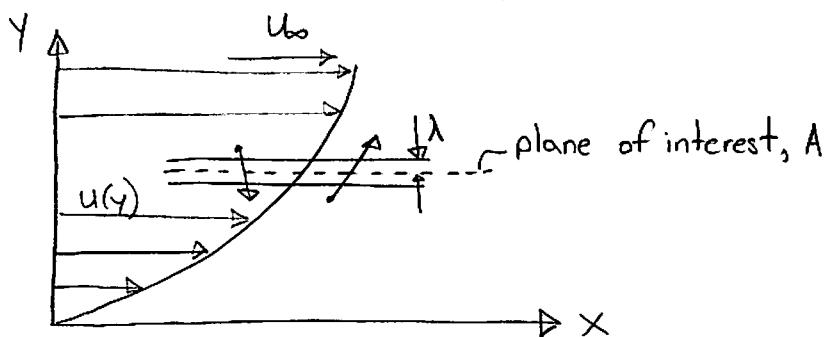
$$\bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{V}}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \frac{\partial^2 \bar{U}}{\partial y^2} - \frac{2}{\partial y} (\bar{U}'\bar{V}')$$

Rewriting this:

$$\boxed{\bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{V}}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial y} \left(U \frac{\partial \bar{U}}{\partial y} - \bar{P} \bar{U}'\bar{V}' \right)}$$

Looks familiar!

OK, now back to our problem. We need to solve for U_L . We can make an analogy to molecular diffusion:



Assuming:

- 1) n molecules per unit volume
- 2) $1/3$ of molecules have velocities along the y -dir. (\bar{V})
- 3) $\frac{1}{6}n$ travel in $+y$ -dir., and $\frac{1}{6}n$ travel in $-y$ -dir. (\bar{V})
- 4) mean velocity \bar{V} in the y -direction for above averages.

Note, molecules will be randomly distributed with more than $\frac{1}{3}n$ having velocity components in y -direction, however their y -velocities will also be randomized and not all \bar{V} . Hence if we average out direction & speed, we will get $\frac{1}{3}n\bar{V}$ in y -dir.

$\frac{1}{6}n\bar{v}$ cross plane A from below

$\frac{1}{6}n\bar{v}$ cross plane A from above

Molecules experienced their last collision (momentum exchange) a distance λ away from the plane A (λ = mean free path).

So an x-momentum balance yields:

$$m_{x\uparrow} = \frac{1}{6}n\bar{v} [m u_x(y-\lambda)] \quad = \text{momentum transported per unit time per unit area across plane upwards}$$

$$m_{x\downarrow} = \frac{1}{6}n\bar{v} [m u_x(y+\lambda)]$$

$$\tau = \frac{1}{6} n \bar{v} m [u_x(y-\lambda) - u_x(y+\lambda)] \Rightarrow \text{Taylor series expand}$$

$$\boxed{\tau = \frac{1}{6} n \bar{v} m \left(-2 \frac{\partial u_x}{\partial y} \cdot \lambda \right)} = -\mu \frac{\partial u}{\partial y} \Rightarrow \text{Shear stress on plane A.}$$

Note here, $n \cdot m = \rho$

$$\boxed{\mu = \frac{1}{3} \rho \bar{v} \lambda} \Rightarrow \text{Molecular Viscosity (for a gas)}$$

Note, before we showed that: $\lambda = \frac{k_B T}{\pi d^2 \rho}$ (ideal gas)

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$$

Interestingly, back substituting \bar{v} & λ into μ

$$\mu = \frac{1}{3} \rho \sqrt{\frac{8k_B T}{\pi m}} \cdot \frac{k_B T}{\pi d^2 \rho} \Rightarrow n = \frac{\rho}{k_B T} \Rightarrow \rho = n \cdot m$$

$$\mu = \frac{1}{3} m \frac{\rho}{k_B T} \cdot \sqrt{\frac{8k_B T}{\pi m}} \cdot \frac{k_B T}{\pi d^2 \rho} = \frac{m}{3\pi d^2} \sqrt{\frac{8k_B T}{\pi m}}$$

$$\boxed{\mu \sim T^{1/2}} \neq f(\rho, P)!$$

Interesting since μ increases at higher temperatures for gas. Opposite for liquids.

So for our turbulent viscosity, we can assume a similar form as our molecular viscosity.

$$\boxed{\text{Molecules Diffusing}} \xrightarrow{\text{Analogy}} \boxed{\text{Eddies Diffusing}}$$

$$u = \frac{1}{3} \rho v \lambda$$

$\uparrow \downarrow$

$$\boxed{u_t \sim \rho u_* l}$$

mean free path analogous to eddy size

$\Rightarrow u_* = \text{friction velocity}$
 $l = \text{mixing length or Eddy size.}$

This approximation is known as Prandtl's Mixing Length model.
Simplest approach.

Doing a wall coordinate nondimensionalization:

$$u^+ = \frac{\bar{u}}{u_*}, \quad v^+ = \frac{\bar{v}}{u_*}$$

$$x^+ = \frac{x u_*}{\nu}, \quad y^+ = \frac{y u_*}{\nu}$$

We can re-cast $C_0 = \rho \frac{\partial \bar{u}}{\partial y} (\nu + \varepsilon)$ as: $(u_* = \left(\frac{C_0}{\rho}\right)^{1/2})$

$$\underbrace{\frac{C_0}{\rho}}_{u_*^2} = \frac{\partial \bar{u}}{\partial y} (\nu + \varepsilon) \Rightarrow \partial \bar{u} = u_* \partial u^+; \quad \partial y = \frac{\partial y^+ \nu}{u_*}$$

$$\underbrace{u_*^2}_{u_*^2} = \frac{\partial u^+}{\partial y^+} \cdot \frac{u_*^2}{\nu} (\nu + \varepsilon)$$

$$\boxed{\left(1 + \frac{\varepsilon}{\nu}\right) \frac{\partial u^+}{\partial y^+} = 1} \Rightarrow u^+(y^+) = \text{velocity distribution near the wall.}$$

But we still need to know ε to solve. Looking back at Prandtl's mixing length model.

$$U_t = \rho U_* l = \rho \varepsilon \Rightarrow \varepsilon = l \left(\frac{C_0}{\rho} \right)^{1/2}$$

need to determine

Prandtl assumed that the size of turbulent eddies cannot be bigger than their distance from the wall, hence:

$$l = K y \equiv \text{mixing length}$$

Back substituting

$$\left(1 + \frac{\varepsilon}{V} \right) \frac{\partial u^+}{\partial y^+} = 1$$

In the viscous sublayer, $V \gg \varepsilon$, $\frac{\varepsilon}{V} \ll 1$

$$\int \frac{\partial u^+}{\partial y^+} = 1$$

$$u^+ = y^+ \equiv \text{Viscous sublayer velocity profile}$$

In the turbulent layer, $\varepsilon \gg V$, $\frac{\varepsilon}{V} \gg 1$

$$\frac{\varepsilon}{V} \frac{\partial u^+}{\partial y^+} = 1 \rightarrow \text{use mixing length model}$$

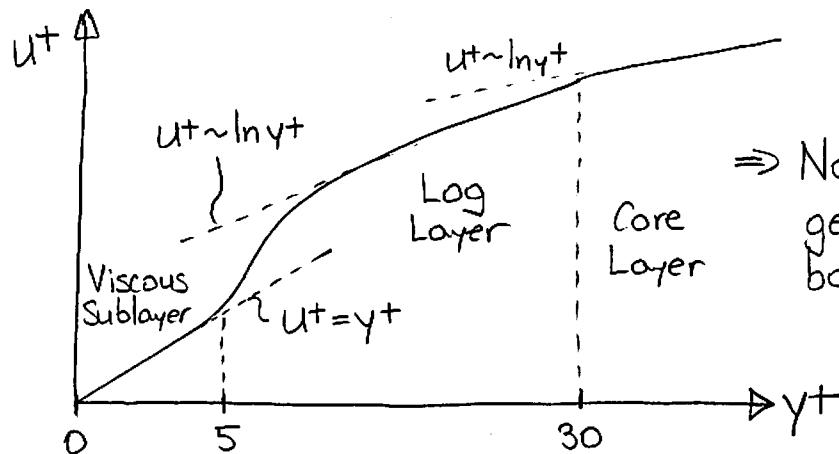
$$\frac{l U_*}{V} \frac{\partial u^+}{\partial y^+} = 1 \Rightarrow l = K y = K \frac{y^+ V}{U_*}$$

$$K y + \frac{V}{U_*} \cdot \frac{U_*}{V} \frac{\partial u^+}{\partial y^+} = 1$$

$$K y + \frac{\partial u^+}{\partial y^+} = 1 \Rightarrow \int \partial u^+ = \int \frac{dy^+}{y^+ K}$$

$$u^+ = \frac{1}{K} \ln y^+ + C \equiv \text{Log-layer velocity profile.}$$

Note, this turbulent b.l. behaviour is also applicable to internal flows.



\Rightarrow Note, applicable to general turbulent boundary layer flow.

Experimentally verified solutions are:

$$\begin{aligned} U^+ &= y^+ ; & 0 < y^+ < 11.6 \\ U^+ &= 2.5 \ln y^+ + 5.5 ; & y^+ > 11.6 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Prandtl \& Taylor}$$

$$\begin{aligned} U^+ &= y^+ & 0 < y^+ < 5 \\ U^+ &= 5 \ln y^+ - 3.05 & 5 < y^+ < 30 \\ U^+ &= 2.5 \ln y^+ + 5.5 & y^+ > 30 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{von Karman (1939)} \\ \text{Result plotted above} \end{array}$$

So to solve for shear stress in turbulent flows, we need to solve for C_0 using our profiles above. Note:

$$C_0 \sim \frac{\partial \bar{U}}{\partial y} \sim \frac{U_\infty}{S} \Rightarrow \text{We need } S \text{ for a turbulent b.l.}$$

We can approximate our piecewise profiles with one good solution: $U^+ = f(y^+)$

$$U^+ = 8.75(y^+)^{1/7} \Rightarrow \text{Prandtl's } \neq \text{ power law}$$

Dimensionalizing our equation ($U^+ = f(y^+)$) for $\bar{U} = U_\infty$ at $y=S$

$$\frac{\bar{U}}{U_*} = f\left(\frac{y U_*}{V}\right) \Rightarrow U_* = \left(\frac{C_0}{\rho}\right)^{1/2}$$

$$\text{For } \bar{U} = U_\infty, y = S \Rightarrow \boxed{\frac{U_\infty}{(C_0/\rho)^{1/2}} = f\left(\frac{S}{V} \left(\frac{C_0}{\rho}\right)^{1/2}\right)}$$

Using Prandtl's $1/7$ power law as f , i.e. $f = 8.75(y^+)^{1/7}$
And applying our momentum integral equation:

$$\frac{d}{dx} \int_0^\infty \bar{U} (U_\infty - \bar{U}) dy = \frac{C_0}{\rho} \quad ①$$

$$\frac{U_\infty}{(C_0/\rho)^{1/2}} = 8.75 \left(\frac{\delta}{V} \left(\frac{C_0}{\rho} \right)^{1/2} \right)^{1/7} \quad ② \Rightarrow \text{Solve for } C_0$$

$$C_0 = 0.0225 \rho U_\infty^2 \left(\frac{\delta U_\infty}{V} \right)^{-1/4} \quad ③$$

Substituting ③ into ① & integrate to $y = S \Rightarrow$ we can now solve.
Complex method since $\bar{U} = f(C_0)$ as well,

$$\frac{S}{x} = 0.37 \left(\frac{U_\infty x}{V} \right)^{-1/5} = \frac{0.37}{Re_x^{0.2}} \quad ④ \Rightarrow \text{Turbulent b.l. thickness.}$$

Combining ④ and ② yields:

$$\boxed{\frac{C_0}{\rho U_\infty^2} = \frac{1}{2} C_{f,x} = 0.0296 \left(\frac{U_\infty x}{V} \right)^{-1/5}}$$

\Rightarrow Local turbulent skin friction coeff.
Flat plate.

$$\boxed{\frac{\overline{C}_0}{\rho U_\infty^2} = \frac{1}{2} \overline{C}_f = 0.037 \left(\frac{U_\infty x}{V} \right)^{-1/5}}$$

\Rightarrow Average turbulent skin friction coeff.
 $10^5 < Re_x < 10^8$

Experiments show that:

$$\boxed{C_{f,x} = 0.37 \left[\log_{10} \left(\frac{U_\infty x}{V} \right) \right]^{-2.584}} \quad 10^5 < Re_x < 10^{10}$$

Note, discrepancy arises between experiment & theory at $Re_x > 10^8$
due to our assumption of $u^+ = f(y^+) \Rightarrow$ Prandtl's $1/7$ Power law.

Turbulent Heat Transfer

If we follow the same procedure as before of time averaging our energy equation:

$$\bar{U} \frac{\partial \bar{T}}{\partial x} + \bar{V} \frac{\partial \bar{T}}{\partial y} = \alpha \nabla^2 \bar{T} - \frac{\partial}{\partial x} (\bar{U}' \bar{T}') - \frac{\partial}{\partial y} (\bar{V}' \bar{T}')$$

↳ Time averaged energy equation for turbulent flow.

For boundary layers: ($U' \sim V'$)

$$\frac{\frac{\partial}{\partial x} (\bar{U}' \bar{T}')}{\frac{\partial}{\partial y} (\bar{V}' \bar{T}')} \sim \frac{\frac{\bar{U}' \bar{T}'}{L}}{\frac{\bar{V}' \bar{T}'}{S_T}} \sim \frac{S_T}{L} \ll 1 \Rightarrow \text{so we can drop the } x \text{ term}$$

Our b.l. energy equation becomes:

$$\begin{aligned} \bar{U} \frac{\partial \bar{T}}{\partial x} + \bar{V} \frac{\partial \bar{T}}{\partial y} &= \underbrace{\alpha \frac{\partial^2 \bar{T}}{\partial y^2}}_{= \frac{1}{\rho C_p} \frac{\partial \bar{T}}{\partial y} (k \frac{\partial \bar{T}}{\partial y} - \rho C_p \bar{V}' \bar{T}')} - \frac{\partial}{\partial y} (\bar{V}' \bar{T}') \\ &= \frac{1}{\rho C_p} \frac{\partial \bar{T}}{\partial y} (k \frac{\partial \bar{T}}{\partial y} - \rho C_p \bar{V}' \bar{T}') \end{aligned}$$

Looks analogous to our shear solution a few pages ago.

Using the same reasoning as before, we can show:

$$-\rho C_p \bar{V}' \bar{T}' = \rho C_p \mathcal{E}_T \frac{\partial \bar{T}}{\partial y} \quad \begin{aligned} &= \text{Eddy Heat Flux} \\ &\mathcal{E}_T \equiv \text{eddy thermal diffusivity} \end{aligned}$$

$$-q''_{app} = k \frac{\partial \bar{T}}{\partial y} - \rho C_p \bar{V}' \bar{T}' = \rho C_p (\alpha + \mathcal{E}_T) \frac{\partial \bar{T}}{\partial y} \quad \begin{aligned} &= \text{Apparent Heat Flux} \end{aligned}$$

We can also define:

$$Pr_T = \frac{\mathcal{E}_T}{\mathcal{E}_T} = \text{turbulent Prandtl \#}$$

The nice thing is we don't have to solve for the heat flux. We can use the Turbulent Colburn Analogy.

$$St_x \cdot Pr^{2/3} = \frac{C_{f,x}}{2} \Rightarrow \text{Colburn Analogy (Did this in Laminar flow)}$$

$$St_x = \frac{Nu_x}{Re_x \cdot Pr} = \frac{\bar{h}x/k}{\cancel{\rho u_\infty x} \cdot \cancel{(0)} \frac{C_p P}{k}} = \frac{\bar{h}}{\cancel{\rho C_p U_\infty}} = \frac{\bar{q}''/\Delta T}{\cancel{\rho C_p U_\infty}}$$

$$Nu_x = \frac{C_{f,x}}{2} Re_x \cdot Pr^{1/3} \Rightarrow \text{We already solved for } C_{f,x}$$

$$C_{f,x} = 0.0592 Re_x^{-1/5} \Rightarrow \text{External turbulent flow over flat plate}$$

Back substituting, we obtain:

$$Nu_x = \frac{0.0592}{2} Re_x^{-1/5} \cdot Re_x \cdot Pr^{1/3}$$

$$Nu_x = 0.029 Re_x^{4/5} Pr^{1/3} \Rightarrow \text{Colburn correlation, } Pr \geq 0.5, Re_x > 5 \times 10^5$$

For the best correlation (most accurate):

$$Nu_x = \frac{\left(\frac{C_{f,x}}{2}\right) \cdot Re_x \cdot Pr}{1 + 12.7 \left(\frac{C_{f,x}}{x}\right)^{1/2} (Pr^{2/3} - 1)} \Rightarrow \begin{aligned} &\text{White correlation} \\ &0.5 \leq Pr \leq 2000 \\ &5 \times 10^5 \leq Re_x \leq 10^7 \end{aligned}$$

Using the Colburn result:

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.037 Re_L^{4/5} Pr^{1/3}, \quad Pr \geq 0.5, \quad Re_x > 5 \times 10^5$$

Note, our correlations are valid for both $T_o = \text{constant}$ and $\dot{q}''|_o = \text{constant}$

For $\dot{q}''|_o = \text{constant}$, only a 4% difference than the results above,
Note also $Nu_x = \frac{\dot{q}''|_x}{k[T_o(x) - T_\infty]} = \text{constant heat flux Nusselt #}$

Side Note: Proof of the Colburn Analogy:

$$\left. \begin{aligned} St &= \frac{h}{\rho C_p U_\infty}, \quad \alpha = \frac{h}{\rho C_p} \\ C_0 &= u \frac{\partial \bar{U}}{\partial y} \\ q''_{\text{ext}} &= h \frac{\partial T}{\partial y} \end{aligned} \right\} \frac{C_0}{q''_{\text{ext}}} = \frac{u}{h} \frac{\partial \bar{U}}{\partial T} \sim \frac{u}{h} \frac{U_\infty}{\Delta T}$$

Here, we know $h = \rho C_p \alpha$, $q''_{\text{ext}} = h \Delta T$ and $C_0 = \frac{1}{2} \rho U_\infty^2 C_{f,x}$.
Back substituting:

$$\frac{C_0}{q''_{\text{ext}}} = \frac{\frac{1}{2} \rho U_\infty^2 C_{f,x}}{h \Delta T} \sim \frac{u U_\infty}{h \Delta T}$$

$$\frac{\frac{1}{2} U_\infty C_{f,x}}{h} \sim \underbrace{\frac{u}{U}}_{Pr} \frac{1}{\rho C_p \alpha} \sim \underbrace{\frac{V}{\alpha}}_{Pr} \cdot \frac{1}{\rho C_p}$$

$$\frac{\frac{1}{2} C_{f,x}}{h} \sim Pr \frac{1}{\rho C_p U_\infty} \Rightarrow \frac{1}{2} C_{f,x} \sim \underbrace{\frac{h}{\rho C_p U_\infty}}_{St} \cdot Pr$$

$$\therefore \boxed{St = \frac{h}{\rho C_p U_\infty} \sim \frac{1}{2} \cdot \frac{C_{f,x}}{Pr}} \Rightarrow \text{QED.}$$

Turbulent Internal Flows

For internal flows, it's a similar procedure as external. Turbulence acts to increase mixing and hence increase shear and heat transfer.

Our time averaged governing equations become (for a tube)

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \bar{v}) = 0 \quad = \text{Mass Conservation}$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial r} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left[r (\nu + \varepsilon) \frac{\partial \bar{u}}{\partial r} \right] \equiv \text{Momentum}$$

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left[r (\alpha + \varepsilon_t) \frac{\partial \bar{T}}{\partial r} \right] \equiv \text{Energy}$$

For turbulent flows:

$$\boxed{\frac{X_{EL}}{D} \approx \frac{X_T}{D} = 10} \Rightarrow X_{EL} \equiv \text{hydrodynamic developing length} \\ X_T \equiv \text{thermal developing length}$$

Note, $X_{EL, \text{laminar}} \gg X_{EL, \text{turbulent}}$ } Better mixing in
 $X_{T, \text{laminar}} \gg X_{T, \text{turb.}}$ turbulent b.l.s.

For fully developed flow, we can use scaling to show:

$$0 = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left[r (\nu + \varepsilon) \frac{\partial \bar{u}}{\partial r} \right] \quad ①$$

Integrating ① from the centerline ($r=0$) to any r :

$$\int_0^r \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} r dr = \int_0^r \partial \left[r (\nu + \varepsilon) \frac{\partial \bar{u}}{\partial r} \right]$$

$$\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} \frac{r^2}{2} = r (\nu + \varepsilon) \frac{\partial \bar{u}}{\partial r} \quad ②$$

$$\text{At the wall: } r=r_0, \varepsilon \ll \nu \Rightarrow \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} \cdot \frac{r_0^2}{2} = \nu \frac{\partial \bar{u}}{\partial r} \Big|_{r_0} \quad ③$$

But remembering from before:

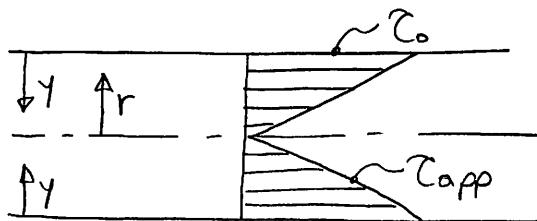
$$\tau_{app} = \rho(v + \epsilon) \frac{\partial \bar{u}}{\partial y} \quad \text{and} \quad \tau_0 = \rho v \frac{\partial \bar{u}}{\partial y} \Big|_{y=0}$$

Here, $y = r_0 - r$ (distance from the wall)

Dividing ② and ③, we obtain: (and noting $r = r_0 - y$)

$$\frac{\frac{1}{\rho} \frac{\partial P}{\partial x} \frac{r}{2}}{\frac{1}{\rho} \frac{\partial P}{\partial x} \cdot \frac{r_0}{2}} = \frac{\tau_{app}}{\tau_0} \Rightarrow \boxed{\frac{\tau_{app}}{\tau_0} = 1 - \frac{y}{r_0}}$$

↳ Fully developed flow momentum eqn.



Very close to the wall ($\frac{y}{r_0} \ll 1$): $\frac{\tau_{app}}{\tau_0} = 1$ or $\tau_{app} = \tau_0 = \text{const}$

We can now use the mixing length model developed previously:
Same result as before!

$$\begin{aligned} u^+ &= y^+ \\ u^+ &= 2.5 \ln y^+ + 5.5 \end{aligned} \quad \left. \begin{aligned} 0 < y^+ < 11.6 \\ y^+ > 11.6 \end{aligned} \right\} \text{Prandtl \& Taylor}$$

Note our model breaks down at the pipe centerline since

$$\frac{\partial u^+}{\partial y^+} \Big|_{y=r_0} = \frac{2.5 v}{r_0 (\tau_0 / \rho)^{1/2}}$$

So people usually use an empirical profile with $\frac{\partial u^+}{\partial y^+} \Big|_{y=r_0} = 0$

$$u^+ = 2.5 \ln \left[\frac{3(1 + r/r_0)}{2[1 + 2(r/r_0)^2]} \cdot y^+ \right] + 5.5$$

⇒ Pipe flow law of the wall

Note, for $y \rightarrow 0$ (wall), $u^+ = 2.5 \ln(y^+) + 5.5$
same as before!

Remembering our definition of friction factor (U = average velocity)

$$f = \frac{C_0}{\frac{1}{2} \rho U^2} ; \quad U = \frac{1}{\pi r_0^2} \int_0^{2\pi} \int_{r_0}^{\infty} \bar{u} r dr d\theta$$

Let's consider Prandtl's $1/7$ power law and solve:

Note, we will assume it holds all the way to the centerline, however it doesn't in real life. It works well at the wall though.

At the centerline, $\bar{u} = U_c$, $y = r_0$

$$U^+ = 8.75 (y^+)^{1/7} ; \quad U^+ = \frac{\bar{u}}{U_*} = \frac{\bar{u}}{(C_0/\rho)^{1/2}} , \quad y^+ = \frac{y U^*}{U}$$

$$U_c^+ = 8.75 (y_c^+)^{1/7} \quad (\text{Apply at the centerline})$$

$$\frac{U_c}{(C_0/\rho)^{1/2}} \approx 8.75 \left[\frac{r_0 (C_0/\rho)^{1/2}}{U} \right]^{1/7} \quad ①$$

Rearranging our friction factor definition:

$$f = \frac{C_0}{\frac{1}{2} \rho U^2} \Rightarrow \left(\frac{C_0}{\rho} \right)^{1/2} = \left(\frac{1}{2} f U^2 \right)^{1/2} = U \left(\frac{f}{2} \right)^{1/2} \quad ②$$

$$(C_0/\rho)^{1/2} = U_* = U \left(\frac{f}{2} \right)^{1/2}$$

We can also say:

$$\frac{\bar{u}}{U_*} \approx 8.75 \left(\frac{y U_*}{U} \right) = 8.75 \left(\frac{y (C_0/\rho)^{1/2}}{U} \right)^{1/7} \quad ③$$

Divide ③ by ①, we obtain:

$$\frac{\frac{\bar{u}}{U_*}}{\frac{U_c}{U_*}} \approx \frac{8.75 \left(\frac{y U_*}{U} \right)^{1/7}}{8.75 \left(\frac{r_0 U_*}{U} \right)^{1/7}} \Rightarrow \frac{\bar{u}}{U_c} = \left(\frac{y}{r_0} \right)^{1/7} \quad ④$$

Back substituting ④ into our definition for U (avg. velocity),

$$\bar{U} = \frac{1}{\pi r_0^2} \int_0^{2\pi} \int_{r_0}^{r_0} \bar{U} r dr d\theta$$

For a pipe, $\int_0^{2\pi} d\theta = 2\pi \Rightarrow \bar{U} = \frac{1}{\pi r_0^2} \int_{r_0}^{r_0} \bar{U} 2\pi r dr$ (no- θ -dependence)

$$\bar{U} = U_c \left(\frac{y}{r_0}\right)^{1/7}$$

$$\bar{U} = \frac{1}{\pi r_0^2} \int_0^{r_0} U_c \left(\frac{y}{r_0}\right)^{1/7} 2\pi r dr \Rightarrow y = r_0 - r$$

$$= \frac{1}{\pi r_0^2} \int_0^{r_0} U_c \left(\frac{r_0 - r}{r_0}\right)^{1/7} 2\pi r dr$$

$$= \frac{2\pi U_c}{\pi r_0^2} \int_0^{r_0} r \left(1 - \frac{r}{r_0}\right)^{1/7} dr = \frac{2U_c}{r_0^2} \int_0^{r_0} \left(r^7 - \frac{r^8}{r_0}\right)^{1/7} dr$$

Solving numerically or analytically, you obtain:

$$\frac{U_c}{U} = \frac{120}{98} \quad (5)$$

Now we can solve for our friction factor by back substituting into (1) and using (2) & (5)

$$\frac{U_c}{U \left(\frac{f}{2}\right)^{1/2}} \approx 8.75 \left[\frac{r_0 \left(U \left(\frac{f}{2}\right)^{1/2}\right)}{V} \right]^{1/7} \Rightarrow \text{Substitute (5) in here}$$

$$\frac{120}{98 \left(\frac{f}{2}\right)^{1/2}} \approx 8.75 \left[\underbrace{\frac{r_0 U}{V}}_{Re_r} \left(\frac{f}{2}\right)^{1/2} \right]^{1/7}$$

$$Re_r = \frac{1}{2} Re_0$$

$$\frac{120 \cdot \sqrt{2}}{98 f^{1/2}} \approx 8.75 \left(\frac{Re_0}{2\sqrt{2}} \cdot f^{1/2} \right)^{1/7} \Rightarrow \frac{0.1979}{f^{1/2}} \approx (0.3536 Re_0 f^{1/2})^{1/4}$$

$$\frac{0.2295}{Re_0^{1/7}} = f^{1/14} \cdot f^{1/2} = f^{8/14} \Rightarrow \text{Raise both sides to power of } \frac{14}{8}$$

$$(0.2295 \cdot Re_0^{-1/7})^{14/8} = f \Rightarrow \boxed{f = 0.078 Re_0^{-1/4}} \Rightarrow Re_0 < 80,000$$

\hookrightarrow Smooth Pipe

Note how close our solution was to Blasius' empirical formula:

$$f \cong 0.079 Re_0^{-1/4} \Rightarrow \text{Blasius}$$

For a more accurate solution using the law at the wall ($U^+ = 2.5 \ln y^+ + 5.5$) instead of the $1/7$ power law:

$$\frac{1}{f^{1/2}} = 1.737 \ln [Re_0 f^{1/2}] - 0.396 \quad \begin{aligned} &\Rightarrow \text{Kármán-Nikuradse} \\ &\Rightarrow Re_0 < 10^6 \\ &\Rightarrow \text{Smooth tubes} \end{aligned}$$

↳ This is the smooth tube solution

Effect of Surface Roughness

Roughness important due to thin nature of laminar sublayer.

Since: $y_{vsl}^+ \sim O(10)$ \Rightarrow For water flow

$$U \sim 10 \text{ m/s}, D \sim 0.01 \text{ cm}^2/\text{s}$$

$$y_{vsl} \sim 0.01 \text{ mm} = 10 \mu\text{m}!$$

So even minute roughness not felt to our touch has big effects on the flow.

For fully rough tubes where roughness size exceeds the order of magnitude of what would have been the viscous sublayer,

$$k_s^+ = \frac{k_s (\tau_0/D)^{1/2}}{D} > O(10) ; k_s = \text{grain or roughness size}$$

Then, Nikuradse showed:

$$f \cong \left(1.74 \ln \left(\frac{D}{k_s} \right) + 2.28 \right)^{-2} \quad \begin{aligned} &\Rightarrow \text{rough pipes (fully rough)} \\ &\Rightarrow \text{Turbulent flow in pipes} \end{aligned}$$

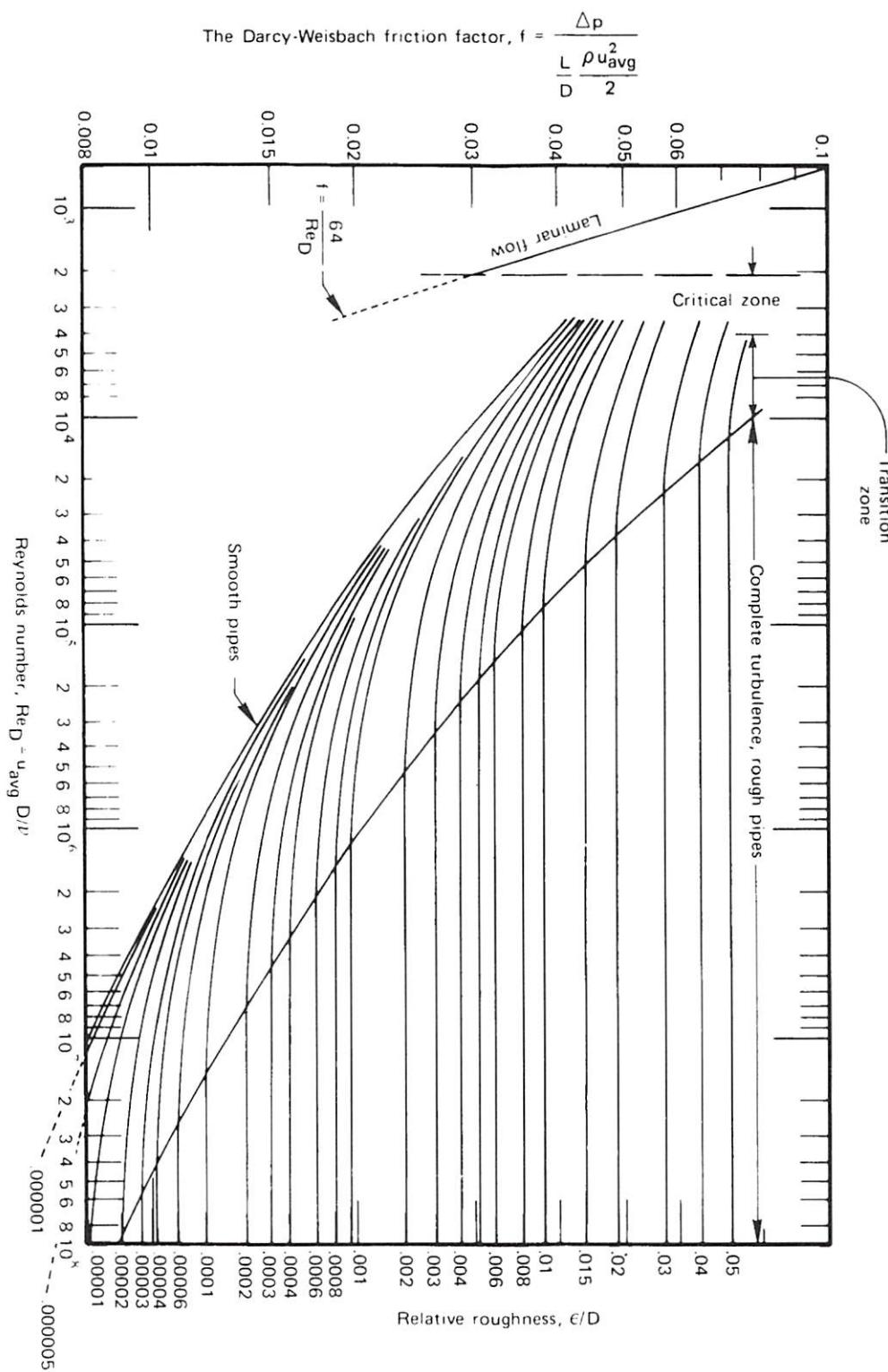


Figure 7.6 Pipe friction factors.

* Adapted from Lienhard & Lienhard, 4th Ed, pp. 361

Turbulent Internal Flow Heat Transfer

Writing out our energy equation for fully developed flow,

$$\rho c_p \bar{U} \frac{\partial \bar{T}}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} (r q''_{app}) ; \quad q''_{app} = -k \frac{\partial \bar{T}}{\partial r}$$

Integrating from 0 to r and 0 to r_0

$$\int_0^r \rho c_p \bar{U} \frac{\partial \bar{T}}{\partial x} r dr = \int_0^r \partial(r q''_{app}) = r q''_{app} \quad (1)$$

$$\int_0^{r_0} \rho c_p \bar{U} \frac{\partial \bar{T}}{\partial x} r dr = r_0 q''_0 \quad (2) \quad (q''_0 = \text{wall heat flux})$$

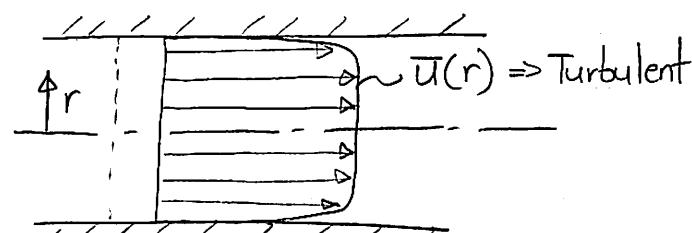
Dividing (1) & (2), we obtain

$$\frac{q''_{app}}{q''_0} = M \left(1 - \frac{r}{r_0}\right)$$

$$M = \frac{\frac{2}{r^2} \int_0^r \bar{U} \frac{\partial \bar{T}}{\partial x} r dr}{\frac{2}{r_0^2} \int_0^{r_0} \bar{U} \frac{\partial \bar{T}}{\partial x} r dr}$$

Note, if we have an x -independent heat-flux, $\frac{\partial \bar{T}}{\partial x} \neq f(r)$
Fully developed flow.

$$M = \frac{\frac{2}{r^2} \int_0^r \bar{U} r dr}{\frac{2}{r_0^2} \int_0^{r_0} \bar{U} r dr} \Rightarrow$$



Since $\bar{U}(r)$ is not a strong function of r (slug flow) in turbulent flows, then:

$$\frac{2}{r^2} \int_0^r \bar{U} r dr \approx \frac{2}{r_0^2} \int_0^{r_0} \bar{U} r dr \Rightarrow M \approx 1$$

Hence:

$$\frac{q''_{app}}{q''_0} = \left(1 - \frac{r}{r_0}\right) \xrightarrow{\text{Same as}} \frac{C''_{app}}{C''_0} = \left(1 - \frac{r}{r_0}\right)$$

$$\frac{C''_{app}}{C''_0} = \left(1 - \frac{r}{r_0}\right)$$

Remembering that: $\frac{C_{app}}{C_0} = \left(1 - \frac{y}{r_0}\right) \Rightarrow \frac{C_{app}}{C_0} = \frac{q''}{q''_0}$ ③

$$C_{app} = (U + p\epsilon) \frac{\partial \bar{U}}{\partial r} = p(U + \epsilon) \frac{\partial \bar{U}}{\partial r} \quad ④$$

$$-q''_{app} = (k + pC_p \epsilon_t) \frac{\partial \bar{T}}{\partial r} = pC_p (\alpha + \epsilon_t) \frac{\partial \bar{T}}{\partial r} \quad ⑤$$

$$\frac{-p(U + \epsilon) \frac{\partial \bar{U}}{\partial r}}{C_0} = \frac{-pC_p (\alpha + \epsilon_t) \frac{\partial \bar{T}}{\partial r}}{q''_0} \Rightarrow \text{Sub } ④ \& ⑤ \text{ into } ③$$

$$\boxed{\frac{U + \epsilon}{C_0} \frac{\partial \bar{U}}{\partial r} = \frac{C_p (\alpha + \epsilon_t)}{q''_0} \frac{\partial \bar{T}}{\partial r}} \quad ⑥$$

Now we can say that the pipe has 2 regions as drawn previously:
 $0 < y < y_1 \Rightarrow U \gg \epsilon$ and $\alpha \gg \epsilon_t$ ($y = r_0 - r$) (wall region)
 $y_1 < y < r_0 \Rightarrow U \ll \epsilon$ and $\alpha \ll \epsilon_t$ (core region - slug flow)

Integrate eq. ⑥ from 0 to y_1 ,

$$\int_0^{y_1} \frac{U + \epsilon}{C_0} d\bar{U} = \int_0^{y_1} \frac{C_p (\alpha + \epsilon_t)}{q''_0} d\bar{T} \quad (\text{since wall region})$$

$$\frac{U}{C_0} \bar{U}_1 = \frac{C_p \alpha}{-q''_0} (\bar{T}_1 - T_0) \quad ⑦ \Rightarrow \bar{U}_1 \text{ and } \bar{T}_1 \text{ are time-averaged quantities at } y_1.$$

Now integrating from y_1 to y_2 where $\bar{U}(y_2) \approx \bar{U}$ and $\bar{T}(y_2) \approx T_m$

$$\frac{\epsilon}{C_0} (U - \bar{U}_1) = \frac{C_p \epsilon_t}{-q''_0} (\bar{T}_m - \bar{T}_1) \quad ⑧$$

Eliminating \bar{U}_1 & \bar{T}_1 from ⑧ by subtracting ⑦ and knowing that:

$$f = \frac{C_0}{\frac{1}{2} p \bar{U}^2} \text{ and } St = \frac{h}{(60 C_p \bar{U})}, \text{ we obtain:}$$

$$\boxed{St = \frac{f/2}{Pr_t + (\bar{U}_1/\bar{U})(Pr - Pr_t)}} \Rightarrow St = f(f, Pr_t, \bar{U}_1/\bar{U})$$

We can see again that $St \sim \frac{f}{2} \Rightarrow$ Colburn Analogy
 Note we don't know y_1 & \bar{U}_1 or E_t .
 Empirical results actually show:

$$St \cdot Pr^{2/3} \approx \frac{f}{2} \Rightarrow \text{Colburn Analogy still holds}$$

$$\Rightarrow Pr > 0.5; f = \frac{C_o}{\frac{1}{2} \rho U^2}$$

To be used in conjunction with the Moody chart to get f .

So for smooth pipes, we already showed that:

$$f = 0.046 Re_0^{-1/5} \quad (\text{Holds for } 2 \times 10^4 < Re_0 < 10^6)$$

$$St \cdot Pr^{2/3} = 0.023 Re_0^{-1/5} \Rightarrow St = \frac{Nu_0}{Re_0 \cdot Pr}$$

$$\frac{Nu_0 \cdot Pr^{2/3}}{Re_0 \cdot Pr} = 0.023 \cdot Re_0^{-1/5}$$

$$Nu_0 = \frac{h D}{k} = 0.023 \cdot Re_0^{4/5} \cdot Pr^{1/3}$$

\Rightarrow Turbulent flow
 $\Rightarrow 2 \times 10^4 < Re_0 < 10^6$
 $\Rightarrow Pr \geq 0.5$

\hookrightarrow here $h = \text{constant}$, so need to use LMTD method: $q = h A_w \Delta T_{LMTD}$

Some other famous ones include:

$$\Delta T_{LMTD} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1 / \Delta T_2)}$$

$$Nu_0 = 0.023 Re_0^{4/5} Pr^n \Rightarrow \begin{cases} n = 0.4 \text{ for fluid being heated} \\ n = 0.3 \text{ for fluid being cooled} \end{cases}$$

\hookrightarrow Dittus-Boelter relation

$$0.7 < Pr < 120$$

$$2500 < Re_0 < 1.24 \times 10^5$$

$$L/D > 60$$

Where temperature influence on properties is significant:

$$Nu_0 = 0.027 Re_0^{4/5} Pr^{1/3} \left(\frac{U}{U_0} \right)^{0.14} \Rightarrow 0.7 < Pr < 16,700$$

$$Re_0 > 10^4$$

$$U_0 \Rightarrow \text{Wall temp}, U \Rightarrow \text{Mean temp}$$

Most accurate:

$$Nu_0 = \frac{(f/2)(Re_0 - 10^3)Pr}{1 + 12.7(f/2)^{1/2}(Pr^{2/3} - 1)} \Rightarrow \begin{cases} \text{For const. } q'' \text{ or } T_0 \\ \Rightarrow \text{Gnielinski Correlation} \\ \Rightarrow \pm 10\% \text{ acc.} \end{cases}$$

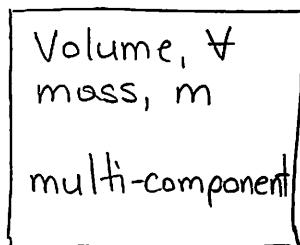
Mass Transfer

Convection heat transfer typically accompanied by mass transfer.
 Atmospheric air \Rightarrow Driven by temperature differences \Rightarrow conveys water vapor

Ocean currents \Rightarrow Driven by $\Delta T \Rightarrow$ convey salt
 Chemical processing \Rightarrow reactive flows

Some basic definitions:

Imagine a container with a multi-component mixture (i.e. Air)



m_i = mass of individual component

$$C_i = \frac{m_i}{V} = \text{concentration of component } i \text{ in the mixture } [\text{kg/m}^3]$$

\uparrow Same thing

$$\rho_i = \frac{m_i}{V} = \text{component density}$$

Since $\sum m_i = m$, then

$$\rho = \sum C_i = \text{aggregate density (what we have been using so far)}$$

Note, the size of our fluid batch is described by extensive properties (m & V). Chemical engineers like to define a batch size by using the molar convention.

$$1 \text{ Mole} = 6.022 \times 10^{23} \text{ elementary entities} = \text{Avogadro's const.}$$

$$M = \text{molar mass} = \text{mass of 1 Mole of a mixture or component } [\text{kg/mol}]$$

So we can now say:

$$\boxed{n = \frac{m}{M}} \quad \text{and} \quad \boxed{n_i = \frac{m_i}{M_i}} \Rightarrow \text{component}$$

\hookrightarrow Mixture

In dimensionless form:

$$\boxed{\phi_i = \frac{m_i}{m}} \Rightarrow \text{Mass fraction} \quad (\sum \phi = 1)$$

or

$$\boxed{x_i = \frac{n_i}{n}}, \quad \boxed{\sum x_i = 1} = \text{Mole Fraction}$$

So we have 2 dimensionless and 1 dimensional way to describe mass transfer. To relate these:

$$\boxed{C_i = \rho \phi_i = \rho \frac{M_i}{M} x_i} \quad \text{where} \quad \boxed{M = \sum M_i x_i}$$

Ideal Gases (high temperature, low pressure)

$$\boxed{PV = mRT} \quad \text{or} \quad \boxed{PV = nRT} \quad \text{where} \quad \boxed{R = 8.314 \text{ J/mol.K}}$$

Here we can define partial pressure as:

$$\boxed{P_i = \frac{R}{M} T} \quad \text{Universal Gas Constant}$$

$$\boxed{P_i V = m_i RT} \quad \text{or} \quad \boxed{P_i V = n_i R T} = \text{pressure one would measure if } i \text{ were to fill volume } V \text{ at temperature } T.$$

Summing over i :

$$\boxed{P = \sum P_i} = \text{Daltons Law} \quad \text{or} \quad \boxed{x_i = \frac{P_i}{P}}$$

Note, this is all valid for equilibrium, homogeneous, mixtures.

Mass Conservation

Applying mass conservation on an elemental control volume for a constituent i :

$$\underbrace{\frac{\partial \rho_i}{\partial t}}_{\text{Accumulation}} + \underbrace{\frac{\partial}{\partial x} (\rho_i u_i) + \frac{\partial}{\partial y} (\rho_i v_i)}_{\text{Net mass flux}} = \underbrace{m_i'''}_{\text{Mass Generation } [\frac{\text{kg}}{\text{m}^3 \cdot \text{s}}]}$$

Note, we get all of our old results back if summing over all constituents and let $m'''_i = 0$.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \sum \rho_i u_i + \frac{\partial}{\partial y} \sum \rho_i v_i = 0$$

\uparrow Has to be true for an equilibrium mixture

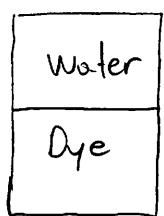
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \Rightarrow \text{Derived in 1'st part of class.}$$

So from this, we can say:

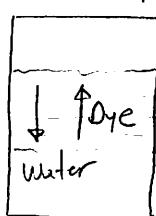
$$U = \frac{1}{\rho} \sum \rho_i u_i ; V = \frac{1}{\rho} \sum \rho_i v_i = \text{Mass averaged velocity}$$

Note, $U \neq u_i$ or $V \neq v_i$. Think of dye diffusing in water

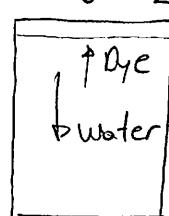
$$t_1 = 0$$



$$t_2 > t_1$$



$$t_3 > t_2$$



$$V_{\text{dye}} = V_1 > 0$$

$$V_{\text{water}} = V_2 < 0$$

$$V = 0$$

While the mass averaged velocity (V) is zero, the velocity of the constituents is not.

$(U_i - U)$ = Diffusion Velocity in x-direction
 $\rho_i(U_i - U)$ = Flow rate per unit area of i in x-direction relative to the bulk motion of the mixture.

$j_{x,i} = \rho_i(U_i - U)$	= Diffusive Flux
$j_{y,i} = \rho_i(V_i - V)$	

Back substituting into our initial constituent mass conservation

$$\frac{\partial \rho_i}{\partial t} + u \frac{\partial \rho_i}{\partial x} + v \frac{\partial \rho_i}{\partial y} = - \frac{\partial j_{x,i}}{\partial x} - \frac{\partial j_{y,i}}{\partial y} + m_i'''$$

Noting that $C_i = \rho_i$ and $\rho = \text{constant}$ (incompressible)

$$\frac{\partial C_i}{\partial t} + u \frac{\partial C_i}{\partial x} + v \frac{\partial C_i}{\partial y} = - \frac{\partial j_{x,i}}{\partial x} - \frac{\partial j_{y,i}}{\partial y} + m_i'''$$

or

$$\boxed{\frac{\partial C_i}{\partial t} = - \nabla \cdot j_i + m_i'''} \Rightarrow \text{Looks Familiar?}$$

The diffusion flux vector j_i is driven by ∇C_i , same as the heat flux vector q'' is driven by ∇T .

In 1855, Fick noticed this and mass transfer was born:
For a 2 component mixture: (dilute approximation, $\rho_{\text{TOT}} = \text{const}$, $[m^2/s]$)

$$\boxed{j_1 = - D_{12} \nabla C_1} ; D_{12} = D_{21} = \overset{\downarrow}{D} = \text{Mass Diffusivity}$$

of component 1 into component 2.

Back substituting j_i into mass conservation above:

$$\boxed{\frac{\partial C}{\partial t} = D \nabla^2 C + m'''} = \text{Dropped the subscript } i$$

↳ Just like heat transfer

Note, in heat transfer we had:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{q'''}{\rho C_p} = \text{Energy Equation}$$

$$q''' = -k \nabla T$$

$$C \rightleftharpoons T, D \rightleftharpoons \alpha$$

So we can solve everything through analogy.

For cartesian

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right) + \dot{m}'''$$

For mass fraction:

$$\frac{\partial \phi}{\partial t} = D \nabla^2 \phi + \frac{\dot{m}'''}{\rho}$$

$$j = -\rho D \nabla \phi$$

For mole fraction:

$$\frac{\partial X}{\partial t} = D \nabla^2 X + \frac{M_i}{M} \frac{\dot{m}'''}{\rho}$$

$$j = -\rho \frac{M_i}{M} D \nabla X$$

$\Rightarrow M_i$ is molar mass of constituent of interest, whose mole fraction is X .

Mass Diffusivities (Binary Mixtures)

See Table 11.1 of pg. 498 in Bejan for $D_{12} = D_{21} = D$ at T_0, P_0

$$\frac{D(T, P)}{D(T_0, P_0)} \approx \left(\frac{T}{T_0} \right)^{1.75} \frac{P_0}{P}$$

*Note, D for liquid mixtures is 10^4 to 10^5 smaller than for gases

Table 11.1 Mass diffusivities of binary gaseous mixtures at atmospheric pressure

Gaseous Mixture	D (m^2/s)	T (K)
Air-acetone	1.09×10^{-5}	273
Air-ammonia	2.80×10^{-5}	298
Air-benzene	0.77×10^{-5}	273
Air-carbon dioxide	1.42×10^{-5} 1.77×10^{-5}	276 317
Air-ethanol	1.45×10^{-5}	313
Air-helium	7.65×10^{-5}	317
Air- <i>n</i> -hexane	0.80×10^{-5}	294
Air-methanol	1.32×10^{-5}	273
Air-naphthalene	5.13×10^{-6}	273
Air-water vapor	2.60×10^{-5} 2.88×10^{-5}	298 313
Ammonia-hydrogen	5.70×10^{-5} 1.10×10^{-4}	263 358
Argon-carbon dioxide	1.33×10^{-5}	276
Argon-hydrogen	8.29×10^{-5}	295
Benzene-hydrogen	4.04×10^{-5}	311
Benzene-nitrogen	1.02×10^{-5}	311
Carbon dioxide-nitrogen	1.67×10^{-5}	298
Carbon dioxide-oxygen	1.53×10^{-5}	293
Carbon dioxide-water vapor	1.98×10^{-5}	307
Cyclohexane-nitrogen	0.73×10^{-5}	288
Helium-methane	6.76×10^{-5}	298
Hydrogen-nitrogen	7.84×10^{-5}	298
Hydrogen-water vapor	9.15×10^{-5}	307
Methane-water vapor	3.56×10^{-5}	352
Nitrogen-water vapor	3.59×10^{-5}	352
Oxygen-water vapor	3.52×10^{-5}	352

*Table adapted from Bejan, Convection Heat Transfer, 4th Edition, pp. 498

Table 11.2 Mass diffusivities of gases and organic solutes at low concentrations in water (dilute aqueous solutions)

Solute	Solvent	D (m^2/s)	T (K)
Acetone	Water	1.16×10^{-9}	293
Air	Water	2.5×10^{-9}	293
Aniline	Water	0.92×10^{-9}	293
Benzene	Water	1.02×10^{-9}	293
Carbon dioxide	Water	1.92×10^{-9}	298
Chlorine	Water	1.25×10^{-9}	298
Ethanol	Water	0.84×10^{-9}	298
Ethylene glycol	Water	1.04×10^{-9}	293
Glycerol	Water	0.72×10^{-9}	288
Hydrogen	Water	4.5×10^{-9}	298
Nitrogen	Water	2.6×10^{-9}	293
Oxygen	Water	2.1×10^{-9}	298
Propane	Water	0.97×10^{-9}	293
Urea	Water	1.2×10^{-9}	293
Vinyl chloride	Water	1.34×10^{-9}	298

*Table adapted from Bejan, Convection Heat Transfer, 4th Edition, pp. 499

Table 11.3 Henry's constant H for several gases in water at moderate pressures

T (K)	H (bar)					
	Air	N_2	O_2	H_2	CO_2	CO
290	6.2×10^4	7.6×10^4	3.8×10^4	6.7×10^4	1.3×10^3	5.1×10^4
300	7.4×10^4	8.9×10^4	4.5×10^4	7.2×10^4	1.7×10^3	6×10^4
320	9.2×10^4	1.1×10^5	5.7×10^4	7.6×10^4	2.7×10^3	7.4×10^4
340	1.04×10^5	1.24×10^5	6.5×10^4	7.6×10^4	3.7×10^3	8.4×10^4

*Table adapted from Bejan, Convection Heat Transfer, 4th Edition, pp. 501

Boundary Conditions

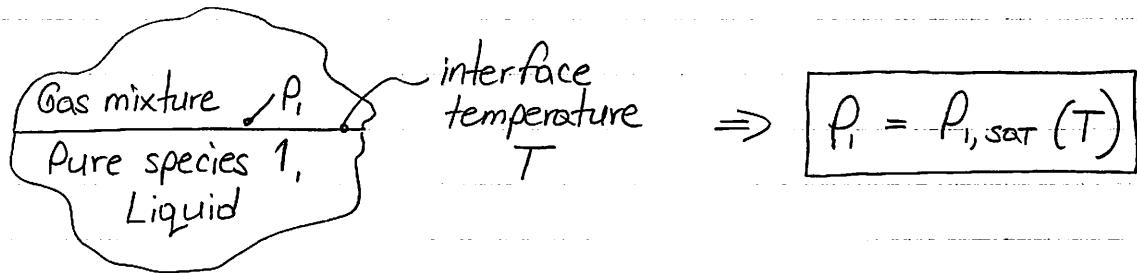
For mass transfer problems, B.C.'s are always applied on the inner side of boundaries facing the domain of mass transfer.

→ concentration

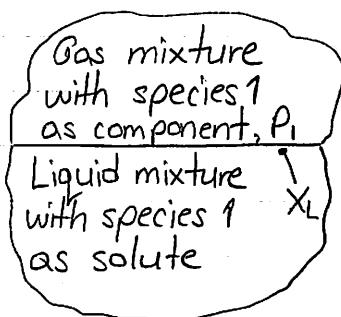
Unlike T, C does not vary continuously across an interface

We can have 3 separate cases to consider:

Case 1: Ideal gas mixture and liquid phase of one component



Case 2: Interface between a liquid mixture and gas mixture



x_L = mole fraction of species 1 on liquid side

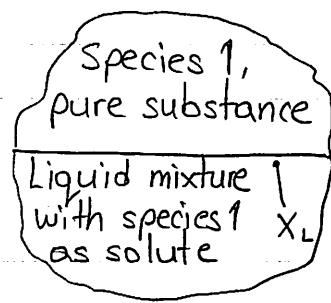
$$x_L = \frac{P_i}{H}$$

P_i = partial pressure of 1.
= Henry's Law
 H = Henry's constant

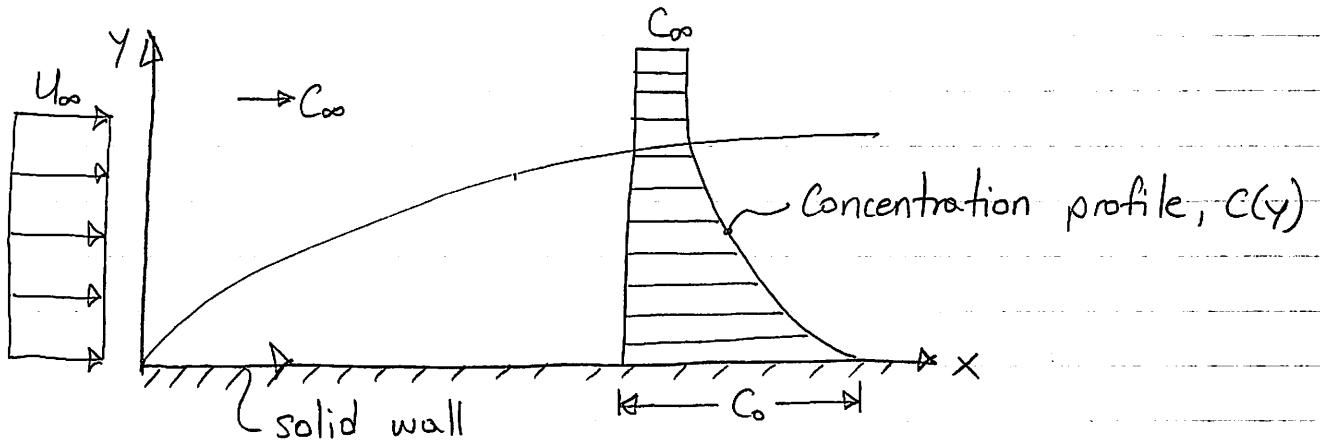
Look up H in Table 11.3 on pg. 188 of notes
Valid for dilute solutions only.

Henry's law also only valid at low pressures ($P_i < 1 \text{ atm}$). At higher pressures, $H = f(P_i)$.

Cases: Interface between a liquid mixture and a pure species 1 (liquid, solid)



⇒ x_L determined by assuming thermodynamic equilibrium at the interface and looking up the solubility of species 1 in the solvent.

Laminar Forced Convection Mass Transfer

Assuming we have a wet wall with humid air at C_∞ flowing past it. Assuming a slender b.l. with $\frac{\partial P_\infty}{\partial x} = 0$ and $T = \text{uniform}$ everywhere.

$$j_o = -D \left(\frac{\partial C}{\partial y} \right) \Big|_{y=0}$$

$$U \frac{\partial C}{\partial x} + V \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} \quad (1)$$

$$\begin{aligned} \text{B.C.'s: } & C = C_0 \text{ at } y=0 \\ & C \rightarrow C_\infty \text{ at } y \rightarrow \infty \end{aligned} \quad \left. \right\} \quad (2)$$

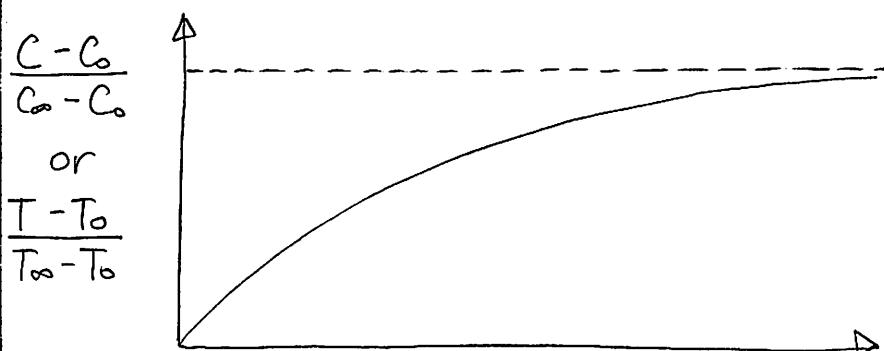
We know (U, V) from the Blasius solution (flow over a flat plate). The mass transfer problem (eq. ① & ②) is analogous to the Pohlhausen solution for heat transfer.

Noting that: $T \rightarrow C$ and $\alpha \rightarrow D$:

$$\left. \frac{\partial C}{\partial y} \right|_{y=0} = (C_\infty - C_0) \left(\frac{U_\infty}{V x} \right)^{1/2} \left\{ \int_0^\infty \exp \left[-\frac{V D}{2} \int f(B) dB \right] d\gamma \right\}^{-1}$$

↳ See page 55 of notes to see how we got this.

Graphically, we can draw this.



$$\eta = \frac{y}{\sqrt{Dx}}, \quad \eta^* = \eta \Pr^{1/3}$$

$$\eta_m^* = \eta Sc^{1/3}$$

For the mass transfer solution:

$$\eta_m^* = \eta Sc^{1/3} \quad \text{where } \boxed{Sc = \frac{V}{D}} = \text{Schmidt \#} \quad V = \text{mixture viscosity}$$

For the thermal solution, we had $\eta^* = \eta \Pr^{1/3}$, so $Sc \leftrightarrow \Pr$. Everything else is analogous!

$$\boxed{h_m = \frac{j_o}{C_0 - C_\infty}} = \text{mass transfer coefficient}$$

All we need to do is replace \Pr with Sc , & h with h_m :

$$\left. \begin{aligned} Nu &= \frac{hx}{k} = 0.332 Re_x^{1/2} \Pr^{1/3} \\ \overline{Nu}_L &= \frac{\overline{h}L}{k} = 0.664 Re_L^{1/2} \Pr^{1/3} \end{aligned} \right\} \text{Heat Transfer}$$

\Downarrow Mass Transfer

$$\left. \begin{aligned} Sh &= \frac{h_m x}{D} = 0.332 Re_x^{1/2} Sc^{1/3} \\ \overline{Sh}_L &= \frac{\overline{h}_m L}{D} = 0.664 Re_L^{1/2} Sc^{1/3} \end{aligned} \right\} \text{Mass Transfer}$$

$Sc > 0.5$

$Sh \equiv$ Sherwood # (Analogous to Nusselt #)

$Nu \rightarrow Sh ; \quad \Pr \rightarrow Sc ; \quad Re_x \rightarrow Re_x ; \quad h \rightarrow h_m ,$

Lastly, we want the mass transfer rate, \dot{m}' (kg of species per second per metered depth):

$$\boxed{\dot{m}' = \overline{h}_m L (C_0 - C_\infty)}$$

$$\boxed{\dot{m} = \overline{h}_m A (C_0 - C_\infty)}$$

$$\overline{h}_m = \frac{1}{L} \int_0^L h_m dx$$

Impermeable Surface Model

Note, our previous solution relies on the Blasius model where we assumed $v|_{y=0} = 0$. For mass transfer problems, this may not be the case.

Let's try some scaling arguments:

$$j_o \sim (C_0 - C_\infty) \frac{D}{\rho} Re_x^{1/2} Sc^n ; \quad n = \frac{1}{3} \text{ if } Sc > 0.5$$

$$n = \frac{1}{2} \text{ if } Sc < 0.5$$

So the velocity scale v associated with the mass flux is:

$$v_o \sim \frac{j_o}{D} \quad ①$$

For our laminar boundary layer, the vertical (transverse) velocity leaving the boundary layer is:

$$v_\infty \sim U_\infty Re_x^{-1/2} \quad (\text{see page 48 of notes})$$

We can say that the solution is valid if $v_o < \sqrt{v_\infty}$

$$\frac{v_o}{v_\infty} \sim \frac{(C_0 - C_\infty) \frac{D}{\rho} Re_x^{1/2} Sc^n}{U_\infty Re_x^{-1/2}} \sim \left(\frac{C_0 - C_\infty}{\rho} \right) \frac{D}{U_\infty x} \cdot Re_x \cdot \left(\frac{U}{D} \right)^n$$

$$\sim \left(\frac{C_0 - C_\infty}{\rho} \right) \frac{D^{1/2}}{U^{1/2}} \cdot \frac{U}{U_\infty x} \cdot Re_x \sim \left(\frac{C_0 - C_\infty}{\rho} \right) \cdot Sc^{-1/2} < 1$$

$$\boxed{\left(\frac{C_0 - C_\infty}{\rho} \right) < Sc^{1/2} < 1}$$

\Rightarrow Valid for small concentration differences.

Colburn Analogy

Since we have an analogy between heat & mass transfer, we can use the Colburn analogy as well for external & internal flows.

$$St_x = \frac{1}{2} C_f x \Pr^{-2/3} \quad (\text{Heat transfer Colburn})$$

Now we need the mass transfer analog to St_x :

$$St_x = \frac{h_x}{\rho C_p U_\infty} = \frac{h_x \alpha}{k U_\infty} = \frac{q'' \alpha}{(T_0 - T_\infty) h U_\infty}$$

For mass transfer: $q'' \rightarrow j_0$

$$T_0 - T_\infty \rightarrow C_0 - C_\infty$$

$$\alpha \rightarrow 0$$

$$h \rightarrow 0$$

$$\frac{q'' \alpha}{(T_0 - T_\infty) h U_\infty} \rightarrow \frac{j_0 D}{(C_0 - C_\infty) D U_\infty} = \frac{h_m (C_0 - C_\infty)}{(C_0 - C_\infty) U_\infty} = \frac{h_m}{U_\infty}$$

$$\therefore \boxed{St_m = \frac{h_m}{U_\infty}} \equiv \text{Local mass transfer Stanton \#}.$$

Now we can say our Colburn analogy becomes:

$$\boxed{St_m \cdot Sc^{2/3} = \frac{C_f x}{2}} \Rightarrow \text{Mass Transfer Colburn analogy}$$

Taking this one step further, & relating our 2 analogies:

$$St_x \Pr^{2/3} = St_m \cdot Sc^{2/3}$$

$$\frac{h}{\rho C_p U_\infty} \cdot \Pr^{2/3} = \frac{h_m}{U_\infty} Sc^{2/3}$$

$$\boxed{\frac{h}{h_m} = (\rho C_p) \left(\frac{\alpha}{D}\right)^{2/3} = \rho C_p Le^{2/3}} \Rightarrow \boxed{Le = \frac{Sc}{Pr} = \frac{\alpha}{D} \equiv \text{Lewis \#}}$$

$\alpha \equiv \text{mixture thermal diff.}$

So by measuring h_m , we can back calculate h !

Turbulent Flow Mass Transfer

Using our same analogy, our solution becomes trivial

$$\overline{Nu}_L = 0.037 \Pr^{1/3} Re_L^{4/5} \quad (\Pr > 0.5, 10^6 < Re_L < 10^8)$$

↓

$$\overline{Sh}_L = \frac{\overline{h}_m L}{D} = 0.037 Sc^{1/3} Re_L^{4/5} \quad (Sc > 0.5, 10^6 < Re_L < 10^8)$$

↳ Flat plate, turbulent, incompressible

For a cylinder in turbulent cross flow:

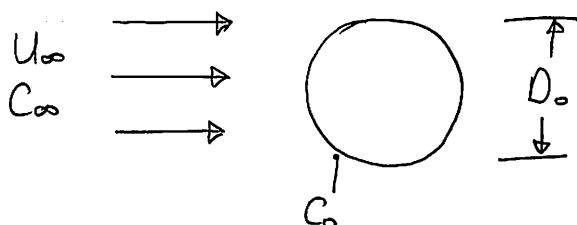
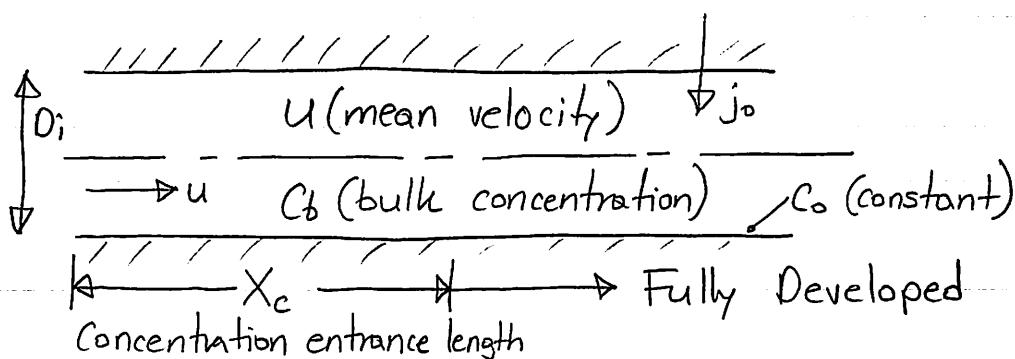
$$\overline{Sh}_{D_0} = \frac{\overline{h}_m D_0}{D} = 0.3 + \frac{0.62 Re_{D_0}^{1/2} Sc^{1/3}}{[1 + (0.4/Sc)^{2/3}]^{1/4}} \cdot \left[1 + \left(\frac{Re_{D_0}}{282,000} \right)^{5/8} \right]^{4/5}$$

$Re_{D_0} Sc > 0.2$

For a sphere in turbulent cross flow:

$$\overline{Sh}_{D_0} = 2 + (0.4 Re_{D_0}^{1/2} + 0.06 Re_{D_0}^{2/3}) Sc^{0.4} \quad (3.5 < Re_{D_0} < 7.6 \times 10^4)$$

Note, our cylinder and sphere cases look like:

Internal Forced Convection Mass Transfer

Note, our analogies still hold for internal flow:

$$\frac{X_c}{D_h} \approx 0.04 Re_{oh} \cdot Sc \Rightarrow \text{Laminar Flow } (Re_{oh} < 2300)$$

$$Re_{oh} = \frac{UD_h}{\nu}$$

$$C_b = \frac{1}{UA} \int_A u C dA = \text{Bulk concentration of the stream}$$

$$Sh_{oi} = \frac{h_m D_i}{D} = 3.66 \Rightarrow D_i = \text{tube inner diameter (pipe)}$$

$D = \text{mass diffusivity}$

$\rightarrow \text{Laminar pipe flow. } (Re_{oh} < 2300)$

For turbulent pipe flow:

$$\frac{X_c}{D_h} \approx 10 \Rightarrow \text{Concentration entrance length in turbulent flow.}$$

Using our mass transfer analogy:

$$Sh_{oi} = 0.023 Re_{oi}^{4/5} Sc^{1/3} \quad (Sc > 0.5, 2 \times 10^4 < Re_{oi} < 10^6)$$

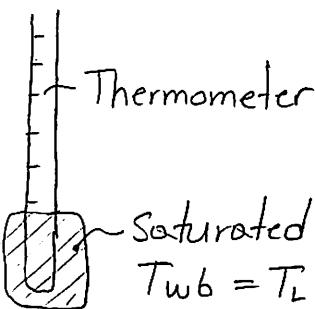
In general, we should always remember:

Heat Transfer	Mass Transfer
$h \left[\text{W/m}^2 \cdot \text{K} \right]$	$h_m \left[\frac{\text{m}}{\text{s}} \right] = \left[\frac{\text{kg}}{\text{m}^2 \cdot \text{s} \cdot \text{kg/m}^3} \right]$
Re (Reynolds)	Re (Reynolds)
$Nu = \frac{hL}{k}$ (Nusselt)	$Sh = \frac{h_m L}{D}$ (Sherwood)
$Nu = f(Re, Pr)$	$Sh = f(Re, Sc)$
$Ra = \frac{g \beta \Delta T L^3}{\nu \alpha}$ (Rayleigh)	$Ram = \frac{g \Delta p L^3}{\mu D}$ (Rayleigh mass transf #)

Wet Bulb Psychrometer (Note, $T_{wb} \neq T_{dew\ point}$)!

A simple device used to measure the relative humidity of air.

$$T_{db} = T_\infty$$



$$T_{wb} = T_L = T_S$$

$\phi = \text{Relative Humidity}$

$$\phi = \frac{P_{H_2O,\infty}}{P_{H_2O,\text{SAT}}(T_\infty)}$$

↳ Need a way to determine $P_{H_2O,\infty}$

Note, we defined on pg. 182 that $C_i = \rho_i$ = component density

So if we do an energy balance on our psychrometer

$$Q_{EVAP} = hA(T_\infty - T_{wb}) \quad ①$$

But we also know that evaporation $\sim h_{fg}$:

$$Q_{EVAP} = m h_{fg} = h_m A (C_{H_2O,S} - C_{H_2O,\infty}) \cdot h_{fg} \quad ②$$

Where h_{fg} = latent heat of phase change

Since $C_{H_2O,S} = \rho_{H_2O,S}$ and $C_{H_2O,\infty} = \rho_{H_2O,\infty}$

And using our ideal gas law for air/water vapor mixture

$$\rho_{H_2O,\infty} = \frac{P_{H_2O,\infty}}{R_{H_2O} \cdot T_\infty} \quad ③$$

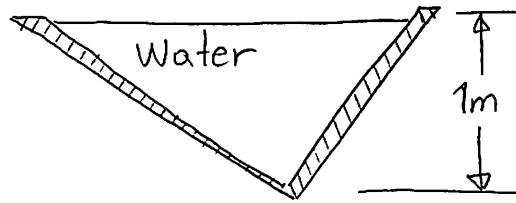
$$\rho_{H_2O,S} = \rho_{H_2O,\text{SAT}}(T_{wb}) = \frac{P_{H_2O,S}}{R_{H_2O} \cdot T_{wb}} \quad ④ \Rightarrow \text{Note the boundary condition.}$$

Rearranging & back substituting ①-④:

$$\underbrace{\left(\frac{h}{h_m}\right) \frac{(T_\infty - T_{wb})}{h_{fg}}}_{R_{H_2O} \cdot C_p L e^{2/3}} = \rho_{H_2O,S} - \rho_{H_2O,\infty} \Rightarrow \begin{array}{l} \text{We know everything} \\ \text{except } P_{H_2O,\infty}. \end{array}$$

$$R_{H_2O} \cdot C_p L e^{2/3} \cdot \frac{(T_\infty - T_{wb})}{h_{fg}} = \frac{P_{H_2O,S}}{T_{wb}} - \frac{P_{H_2O,\infty}}{T_\infty} \Rightarrow \text{Can solve explicitly for } P_{H_2O,\infty} \text{ & } \phi. \quad (196)$$

Ex #1 A channel 25m long & 1m deep used for storage of water. Water & surroundings are at 25°C and RH = φ = 50%.



- a) If air moves at 5m/s along the channel length, determine the rate of water loss due to evaporation.

First we need to determine the flow regime:

$$Re_L = \frac{U_\infty L}{V_{air}} = \frac{(5 \text{ m/s})(25 \text{ m})}{(15.66 \times 10^{-6} \text{ m}^2/\text{s})} = 7.98 \times 10^6 > 5 \times 10^5$$

Turbulent Flow

We can now use our analogies to solve. (Turbulent correlations)

$$\overline{Nu}_L = 0.037 Pr^{1/3} Re_L^{4/5} \quad (Pr \geq 0.5, 10^6 < Re_L < 10^8)$$

$$\overline{h} = \frac{k}{L} \cdot 0.037 Pr^{1/3} Re_L^{4/5} = \frac{(0.0267 \text{ W/m}\cdot\text{K})}{25 \text{ m}} \cdot 0.037 (0.69)^{1/3} \cdot (7.98 \times 10^6)^{4/5}$$

$$\overline{h} = 11.6 \text{ W/m}^2 \cdot \text{K}$$

We know that $\dot{m} = \overline{h}_m A (\rho_s - \rho_\infty)$

Looking up our water vapor densities:

$$\begin{aligned} \xrightarrow{5 \text{ m/s}} \rho_s &= \rho_{\text{sat}}(25^\circ\text{C}) \Rightarrow \rho_s = 0.02282 \text{ kg/m}^3 \\ \text{Water @ } 25^\circ\text{C} \end{aligned}$$

$$\rho_\infty = 0.5 \rho_{\text{sat}}(25^\circ\text{C}) = 0.01141 \text{ kg/m}^3$$

Now we can solve for \overline{h}_m

We showed before that:

$$\frac{\bar{h}}{h_m} = \overbrace{\rho C_p}^{\text{Remember, } \rho \text{ & } C_p \text{ are our mixture properties (air)}} L_e^{2/3}; \quad L_e = \frac{Sc}{Pr}; \quad h_{fg} = 2.442 \times 10^6 \text{ J/kg}$$

$$Sc = \frac{V}{D} \Rightarrow D_{\text{H}_2\text{O-Air}} = 22 \times 10^{-6} \text{ m}^2/\text{s} \quad (\text{From Table II.1, pg. } 187)$$

$$Sc_{\text{Air}} = \frac{15.66 \times 10^{-6} \text{ m}^2/\text{s}}{22 \times 10^{-6} \text{ m}^2/\text{s}} = 0.7118$$

$$L_e = \frac{Sc}{Pr} = \frac{0.7118}{0.69} = 1.0316$$

Back substituting into our analogy

$$\bar{h}_m = \frac{\bar{h}}{\rho C_p L_e^{2/3}} = \frac{11.6 \text{ W/m}^2\text{K}}{(1.177 \text{ kg/m}^3)(1005 \text{ J/kg}\cdot\text{K})(1.0316)^{2/3}}$$

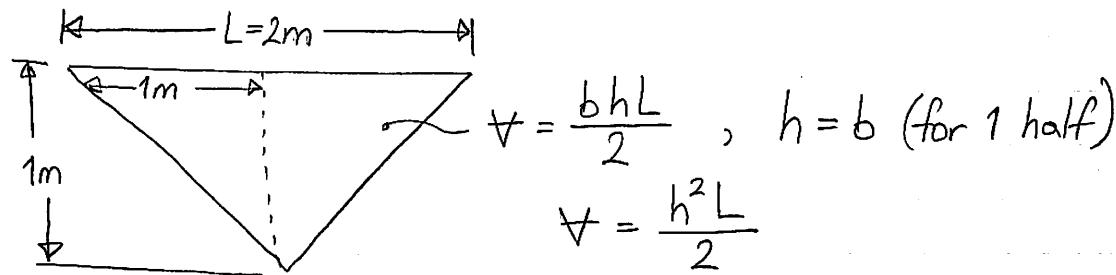
$$\bar{h}_m = 0.007675 \text{ m/s}$$

Now we can calculate \dot{m} and our mass loss:

$$\dot{m} = \bar{h}_m A (\rho_s - \rho_\infty) = (0.007675 \text{ m/s})(50 \text{ m}^2)(0.02282 - 0.01141)$$

$$\boxed{\dot{m} = 0.00438 \text{ kg/s} = 15.76 \text{ kg/hour}}$$

- b) Obtain an expression for the rate of water depth decrease due to evaporation. How long would it take to empty the tank due to evaporation?



$\rightarrow p_{\text{water}}$ since pure water in the tank.

$$\rho_w \frac{\partial h}{\partial t} = \dot{m}$$

$$\frac{\dot{m}}{\rho_w} = \frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{h^2 L}{2} \right) = \frac{1}{2} \cdot \frac{L}{2} \frac{\partial}{\partial t} h^2 = \frac{1}{2} \cdot \frac{L}{2} \cdot 2h \cdot \frac{\partial h}{\partial t}$$

$$\int_0^H h dh = \int_0^t \frac{2\dot{m}}{\rho_w L} dt$$

$$\frac{H^2}{2} = \frac{2\dot{m}}{\rho_w L} t$$

$$\boxed{t = \frac{H^2 \rho L}{2\dot{m}}} \Rightarrow t = \frac{(1m)^2 (1000.0 \text{ kg/m}^3) (25 \text{ m})}{(2)(0.0052 \text{ kg/s})}$$

$$\boxed{t = 27.82 \text{ days}}$$

Note, this is incorrect since we can't assume $\dot{m} = \text{constant}$.
The right way to do it is:

$$\dot{m} = \overline{h_m} (2h)L(p_s - p_\infty)$$

$$\rho \frac{\partial h}{\partial t} = \rho \frac{\partial}{\partial t} \frac{h^2}{2} \cdot L = \overline{h_m} (2hL)(p_s - p_\infty)$$

$$\rho (2h) \frac{1}{2} \frac{\partial h}{\partial t} \cdot L = \overline{h_m} (2hL)(p_s - p_\infty)$$

$$\frac{dh}{dt} = \frac{2\overline{h_m}(p_s - p_\infty)}{\rho}$$

$$h(t) = \frac{2\overline{h_m}(p_s - p_\infty)}{\rho} \cdot t + C$$

$$h(0) = H \Rightarrow C$$

$$h(t) = H + \frac{2\overline{h_m}(p_s - p_\infty)t}{\rho} \Rightarrow \text{Let } h=0 \text{ (when dry)}$$

$$0 = H + \frac{2\overline{h_m}(p_s - p_\infty)}{\rho} t$$

$$\boxed{t = 101.23 \text{ days}}$$